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# Differential Geometry II
# Exercise Sheet no. 3

## Exercise 1

Let $\mathcal{H}_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$ and $\Gamma := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$.

i) Show that $\mathcal{H}_3$ and $\Gamma$ are Lie groups. Does $\mathcal{H}_3$ admit a bi-invariant Riemannian metric?

ii) Show that $\Gamma$ acts on $\mathcal{H}_3$ by left multiplication and this action is free and proper.

iii) Consider the following action of $\mathbb{R}$ on $\mathcal{H}_3$:

$$\mathbb{R} \times \mathcal{H}_3 \to \mathcal{H}_3, \quad \left( \tilde{z}, \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) \mapsto \begin{pmatrix} 1 & x & z + \tilde{z} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that this action descends to an action of $\mathbb{Z}\backslash\mathbb{R}$ on the quotient $\Gamma\backslash\mathcal{H}_3$ and the quotient manifold obtained by this action is the 2-dimensional torus.

## Exercise 2

Let $S^{4n+3} \subset \mathbb{H}^{n+1}$ be the unit sphere in the $(n+1)$-dimensional quaternionic vector space.

i) Show that $S^3 \subset \mathbb{H}$ acts smoothly, freely and properly on $S^{4n+3}$.

ii) Give an atlas for the quotient manifold $\mathbb{H}\mathrm{P}^n := S^3\backslash S^{4n+3}$. The manifold $\mathbb{H}\mathrm{P}^n$ is called the $n$-dimensional quaternionic projective space.

## Exercise 3

i) Determine the Lie bracket $[\cdot,\cdot]$ on $\mathfrak{gl}(n,\mathbb{R})$, the Lie algebra of the general linear group $GL(n,\mathbb{R})$.

ii) For any Lie group $G$ with adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, let $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ denote the differential of Ad at the unit element of $G$, $\mathrm{ad} := \mathrm{d}_{\mathbb{1}}\mathrm{Ad}$.
Show that for $GL(n,\mathbb{R})$, the map ad is given by $\mathrm{ad}(X)(Y) = [X,Y]$, for all $X, Y \in \mathfrak{gl}(n,\mathbb{R})$.

iii) Let $X \in \mathfrak{gl}(n,\mathbb{R})$, $\widetilde{X}$ the corresponding left-invariant vector field on $GL(n,\mathbb{R})$ and $\gamma : \mathbb{R} \to GL(n,\mathbb{R})$ be a curve with $\gamma(0) = \mathbb{1}_n$, $\dot\gamma(t) = \widetilde{X}_{\gamma(t)}$. Show that $\gamma(t) = \sum_{n=0}^{\infty} \frac{1}{n!}(tX)^n$.

*Hand in the solutions on* ***Monday, May 6, 2013*** *before the lecture.*