Universität Regensburg, Mathematik SoSe 2013
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Differential Geometry II
Exercise Sheet no. 4

Exercise 1

Let $\pi : \overline{M} \to M$ be a covering of the manifold M, and let g be a Riemannian metric on M. We equip $\overline{M}$ with the metric $\pi^* g$ defined as

$$\pi^* g_p(X,Y) := g_{\pi(p)}((\mathrm{d}_{\pi(p)}\pi)(X), (\mathrm{d}_{\pi(p)}\pi)(Y)), \quad \forall p \in \overline{M}, \forall X, Y \in T_p\overline{M}. \tag{1}$$

i) Show that if M is compact, then $(\overline{M}, \pi^* g)$ is complete.

ii) Is it still true that $(\overline{M}, \pi^* g)$ is complete when $\pi : \overline{M} \to M$ is only locally diffeomorphic and surjective?

Exercise 2

Let $\pi : \overline{M} \to M$ be a surjective map which is locally diffeomorphic and let g, resp. $\pi^* g$ be Riemannian metrics on M, resp. $\overline{M}$, that are related by (1). We assume that $(\overline{M}, \pi^* g)$ is complete. Show that:

i) (M, g) is also complete.

ii) The map π is a covering. Hint: Use the Hopf-Rinow Theorem.

Exercise 3

Let G be a Lie group, let g a bi-invariant Riemannian metric on G, and let $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ be the map introduced in Exercise 3, ii) on Sheet no. 3.

i) Show that the map ad takes values into the skew-symmetric endomorphisms of $(\mathfrak{g} = T_{\mathbb{1}}G, g_{\mathbb{1}})$. Moreover, one can show that $\mathrm{ad}(X)(Y) = [X,Y]$, for all $X, Y \in \mathfrak{g}$ (we assume this result, it is not part of the exercise to prove it).

ii) Use i) and the Koszul formula to show that the Levi-Civita connection of g is given by $\nabla_X Y = \frac{1}{2}[X,Y]$, for all left-invariant vector fields X, Y.

iii) (Bonus points) Show that the sectional curvature of g is nonnegative. (Hint: First compute the Riemannian curvature tensor using ii): $R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$, for all left-invariant vector fields X, Y, Z. Use also the Jacobi identity).

*Hand in the solutions on **Monday, May 13, 2013** before the lecture.*