Universität Regensburg, Mathematik
Prof. Dr. Bernd Ammann
Dr. Mihaela Pilca

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Differential Geometry II
Exercise Sheet no. 5

Exercise 1

Let $S^3 \subset \mathbb{H}$ be the unit sphere in the quaternion algebra. Consider the following map:

$$\theta : S^3 \times S^3 \to \mathrm{Aut}(\mathbb{H})$$
$$(z, w) \mapsto (q \mapsto zq\overline{w}).$$

i) Show that θ defines a smooth action of $S^3 \times S^3$ on $\mathbb{H}$, which preserves the standard norm on $\mathbb{H} \cong \mathbb{R}^4$.

ii) Compute the kernel of θ.

iii) Show that the differential of θ at the identity element is bijective.

iv) Conclude that θ is the universal covering of $\mathrm{SO}(4)$.

Exercise 2

Let $\mathbb{Z}$ act on $\mathbb{R}^n$ by $k \cdot x := 2^k x$, for $k \in \mathbb{Z}$, $x \in \mathbb{R}^n$.

i) Is this action proper on $M_1 := \mathbb{R}^n$, on $M_2 := \mathbb{R}^n \setminus \{0\}$, on $M_3 := (0, \infty) \times (0, \infty) \times \mathbb{R}^{n-2}$?

ii) Are the quotients $\mathbb{Z} \backslash M_i$ Hausdorff? Are they compact?

Exercise 3

For $0 < m < n$, let $G(m, n)$ be the set of all m-dimensional subspaces in $\mathbb{R}^n$. Show that $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{O}(n, \mathbb{R})$ act transitively on $G(m, n)$. Determine the isotropy groups of $\mathbb{R}^m \times \{0\}$ for both actions, and write $G(m, n)$ as homogeneous space G/H where $G = \mathrm{GL}(n, \mathbb{R})$ or $G = \mathrm{O}(n, \mathbb{R})$.
What is the interpretation of

i) $\mathrm{O}(n, \mathbb{R})/(\mathrm{O}(m, \mathbb{R}) \times \mathrm{O}(n - m, \mathbb{R}))$,

ii) $\mathrm{SO}(n, \mathbb{R})/(\mathrm{SO}(m, \mathbb{R}) \times \mathrm{SO}(n - m, \mathbb{R}))$,

iii) $\mathrm{GL}_+(n, \mathbb{R})/(\mathrm{GL}_+(m, \mathbb{R}) \times \mathrm{GL}_+(n - m, \mathbb{R}))$,

iv) $\mathrm{GL}(n, \mathbb{R})/(\mathrm{GL}(m, \mathbb{R}) \times \mathrm{GL}(n - m, \mathbb{R}))$.

Hint: Be cautious with the isotropy group of $\mathrm{GL}(n, \mathbb{R})$*, and its relation to iii) and iv).*

Hand in the solutions on ***Monday, May 20, 2013*** *before the lecture.*