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# Differential Geometry II
# Exercise Sheet no. 7

**Exercise 1**

The *Killing form* of a Lie algebra $\mathfrak{g}$ is the function defined by:

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad B(X, Y) := \operatorname{tr}(\operatorname{ad}(X) \circ \operatorname{ad}(Y)).$$

Show the following properties of the Killing form:

i) $B$ is a symmetric bilinear form on $\mathfrak{g}$.

ii) If $\mathfrak{g}$ is the Lie algebra of the Lie group $G$, then $B$ is Ad-invariant:

$$B(\operatorname{Ad}(\sigma)X, \operatorname{Ad}(\sigma)Y) = B(X, Y), \quad \forall \sigma \in G, \forall X, Y \in \mathfrak{g}.$$

Hint: Show first that if $\alpha$ is an automorphism of $\mathfrak{g}$, i.e. a linear isomorphism $\alpha$ satisfying $\alpha([X, Y]) = [\alpha(X), \alpha(Y)]$ for all $X, Y \in \mathfrak{g}$, then $\operatorname{ad}(\alpha(X)) = \alpha \circ \operatorname{ad}(X) \circ \alpha^{-1}$, for any $X \in \mathfrak{g}$.

iii) For each $Z \in \mathfrak{g}$, $\operatorname{ad}(Z)$ is skew-symmetric with respect to $B$:

$$B(\operatorname{ad}(Z)X, Y) = -B(X, \operatorname{ad}(Z)X), \forall X, Y \in \mathfrak{g}.$$

**Exercise 2**

Let $(M, g)$ be a Riemannian manifold of constant sectional curvature $\kappa$ and let $\gamma : [0, \ell] \to M$ be a geodesic parametrized by arc-length. Let $J$ be a vector field along $\gamma$, normal to $\gamma'$.

i) Show that the Jacobi equation can be written as $J'' + \kappa J = 0$.

ii) Let $V$ be a parallel unit vector field along $\gamma$ normal to $\gamma'$. Determine the Jacobi vector field $J$ satisfying the initial conditions $J(0) = 0$ and $J'(0) = V(0)$.

**Exercise 3**

i) Let $(M, g)$ be a Riemannian manifold and $\gamma : I \to M$ a geodesic. Show that if $M$ is 2-dimensional, then the relation for points of $\gamma$ to be conjugated to each other along $\gamma$ is transitive. More precisely, for any $t_i \in I$, $i = 1, 2, 3$, such that $\gamma(t_1)$ is conjugated to $\gamma(t_2)$ and $\gamma(t_2)$ is conjugated to $\gamma(t_3)$, it follows that $\gamma(t_1)$ is conjugated to $\gamma(t_3)$.

ii) Show that the statement in i) is not true for higher dimensions, by considering for instance the Riemannian manifold $(S^2 \times S^2, g_{std} \oplus g_{std})$, that is the Riemannian product of two spheres with the standard metric and the following geodesic $\gamma(t) = (\cos(t), 0, \sin(t), \cos(\pi t), 0, \sin(\pi t)) \in S^2 \times S^2 \subset \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$.

*Hand in the solutions on **Monday, June 3, 2013** before the lecture.*