Universität Regensburg, Mathematik SoSe 2013
Prof. Dr. Bernd Ammann 10.06.2013
Dr. Mihaela Pilca

Differential Geometry II
Exercise Sheet no. 9

Exercise 1
Let (M, g) be a connected, complete and simply-connected Riemannian manifold with sectional curvature $K \leq 0$. Show that there is a unique geodesic between any two points on M. Hint: use Cartan-Hadamard Theorem.

Exercise 2
Let M be a connected manifold and $p \in M$. We consider the map defined in the lecture between the fundamental group of M and the set of free homotopy classes of loops:

$$F : \pi_1(M, p) \to \pi_o\mathcal{L}(M),$$
$$[\gamma] \mapsto [\gamma]_{\text{free}}.$$

Show the following:

i) F is surjective.

ii) F induces a well-defined map on the set of conjugacy classes in $\pi_1(M, p)$, that is $[\gamma\tau\gamma^{-1}]_{\text{free}} = [\tau]_{\text{free}}$, for any $\gamma, \tau \in \pi_1(M, p)$.

iii) The map induced by F on the set of conjugacy classes in $\pi_1(M, p)$ is injective.

Exercise 3
We consider the Hopf fibration and the Fubini-Study metric on $\mathbb{C}P^n$ introduced in Exercise 2, (iii) on Sheet no. 8. We use the same notation as in this exercise, and again X^v is the vertical part of X. The vertical vectors of the Hopf fibration in the point $z \in S^{2n+1}$ are of the form λiz, $\lambda \in \mathbb{R}$.
For $X, Y \in \mathbb{C}^{n+1}$, we define $\langle X, Y\rangle_{\mathbb{C}} := \sum_{j=1}^{n+1} X_j\overline{Y}_j$ and $\langle X, Y\rangle_{\mathbb{R}} := \mathrm{Re}(\sum_{j=1}^{n+1} X_j\overline{Y}_j)$.
Then it holds $\langle X, Y\rangle_{\mathbb{C}} = \langle X, Y\rangle_{\mathbb{R}} + i\langle X, iY\rangle_{\mathbb{R}}$. Show the following:

i) For any $\widetilde{X}_0 \in \mathbb{C}^{n+1}$, the map $w \mapsto \widetilde{X}_w := \widetilde{X}_0 - \langle \widetilde{X}_0, w\rangle_{\mathbb{C}} w$ is a well-defined vector field on S^{2n+1}.

ii) $\widetilde{X}$ is horizontal everywhere.

iii) Each point $p \in \mathbb{C}P^n$ admits an open neighborhood U and a smooth map $f : \pi^{-1}(U) \to S^1$, such that $f(\lambda z) = \lambda f(z)$, for all $z \in \pi^{-1}(U)$ and $\lambda \in S^1$.

iv) $f\widetilde{X}$ is a horizontal lift of a vector field $X \in \Gamma(TU)$.

v) For a fixed $z \in S^{2n+1}$ assume that $\langle \widetilde{X}_0, z\rangle_{\mathbb{C}} = \langle \widetilde{Y}_0, z\rangle_{\mathbb{C}} = 0$. For the Levi-Civita connection ∇ of S^{2n+1} it holds:

$$\nabla_{\widetilde{Y}_w} \widetilde{X}_w|_{w=z} = -(\mathrm{Im}(\langle \widetilde{X}_0, \widetilde{Y}_0\rangle_{\mathbb{C}}))iz$$

vi) Choose f such that $f(z_0) = 1$ for a $z_0 \in \pi^{-1}(p)$. Conclude that $[f\widetilde{Y}, f\widetilde{X}]^v|_{z_0} = -2(\mathrm{Im}\langle \widetilde{X}_0, \widetilde{Y}_0\rangle_{\mathbb{C}})iz_0$.

vii) The sectional curvature K of $\mathbb{C}P^n$ satisfies: $1 \leq K \leq 4$. For which planes is $K = 4$ and for which planes is $K = 1$?

Hand in the solutions on ***Monday, June 17, 2013*** *before the lecture.*