Universität Regensburg, Mathematik SoSe 2013
Prof. Dr. Bernd Ammann 17.06.2013
Dr. Mihaela Pilca

Differential Geometry II
Exercise Sheet no. 10

Exercise 1

Determine $\mathcal{C}_p^{\tan}M$, and $\mathcal{C}_pM$ for

(a) $M = \mathbb{R}^2/\Gamma$, where Γ is the subgroup of $\mathbb{R}^2$ generated by $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\2\end{pmatrix}$, and $p := [0]$.

(b) $M = \mathbb{R}P^m = S^m/\{\pm 1\}$ with the quotient metric, and $p := [e_1]$.

Exercise 2

Let $M = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2 = e^{-z^2}\}$. Show that M is a smooth surface, and that M is complete, $\mathrm{vol}(M) < \infty$, $\mathrm{injrad}(M) = 0$, $\mathrm{diam}(M) = \infty$.

Exercise 3

Let M be a complete connected Riemannian manifold, $p \in M$ fixed. We define $\mathrm{diam}\, M := \sup\{d(x,y) \,|\, x,y \in M\}$. Show

(a) $\mathrm{diam}\, M = \sup_{X \in SM} s(X)$

(b) $\mathrm{injrad}(p) = \min_{X \in S_pM} s(X)$

(c) $\mathrm{injrad}(M) = \inf_{X \in SM} s(X)$

(d) $\sup_{X \in SM} s(X) = \infty$ if and only if there is for all $p \in M$ an $X \in S_pM$ with $s(X) = \infty$.
Hint: Use Exercise no. 3 on Sheet no. 9 of Differential Geometry I

(e) Give an example of a complete Riemannian manifold such that $\sup_{X \in S_pM} s(X)$ depends on p.

Exercise 4

We consider $S^3 \subset \mathbb{C}^2$ endowed with the standard metric, and $\Gamma := \{1, i, -1, -i\}$ which acts freely und isometrically on S^3. Let $M := S^3/\Gamma$, $\pi : S^3 \to M$ the corresponding projection and $p := \pi(e_1) = e_1 \mod \Gamma \in M$. Show that for the cut locus $\mathcal{C}_p$ the following holds:

$$\mathcal{C}_p = \{\pi(x) \,|\, x \in S^3 \text{ with } d(x,e_1) = d(x, ie_1)\}$$
$$= \left\{ \pi\left(\frac{(1+i)r}{\sqrt{2}} e_1 + v e_2\right) \,\middle|\, r \in [0,1], \quad v \in \mathbb{C} \text{ with } r^2 + |v|^2 = 1 \right\}.$$

Answer without justification: Where are the minima and maxima of the function $s : S_pM \to (0,\infty)$?
Bonus question: Where is $\mathcal{C}_p$ a smooth hypersurface and where not?

Hand in the solutions on ***Monday, June 24, 2013*** *before the lecture.*