Universität Regensburg, Mathematik SoSe 2013
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# Differential Geometry II
# Exercise Sheet no. 12

**Exercise 1**
Let $(M, g)$ be a Riemannian manifold, whose sectional curvature $K$ satisfies the inequalities:

$$0 < L \leq K \leq H,$$

for some positive constants $L$ and $H$. For a geodesic $\gamma : [0, \ell] \to M$, parametrized by arclength, we define

$$d := \min\{t > 0 \,|\, \gamma(t) \text{ is conjugated to } \gamma(0) \text{ along } \gamma|_{[0,t]}\}.$$

Show

$$\frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}}.$$

*Hint: Use the First Rauch Comparison Theorem.*

**Exercise 2**
Let $(M, g)$ be a complete Riemannian manifold with sectional curvature $K \geq 0$. Let $\Gamma$ be a discrete group without 2-torsion (*i.e.* $\gamma^2 \neq e$, for any $\gamma \in \Gamma \setminus \{e\}$, where $e$ is the identity element of $\Gamma$), acting isometrically, freely and properly on $M$. For a point $p \in M$, let $\gamma_0 \in \Gamma$ be an element with $d(p, \gamma_0 p) = \min\limits_{\gamma \in \Gamma \setminus \{e\}} d(p, \gamma p)$.
We choose a minimal geodesics $c_1$ which connects $p$ to $\gamma_0 p$, and a geodesic $c_2$ which connects $p$ to $\gamma_0^{-1} p$. Show that $c_1$ and $c_2$ form at $p$ an angle $\alpha \geq \frac{\pi}{3}$.

**Exercise 3**
Let $(M, g)$ be a complete Riemannian manifold with sectional curvature $K \geq 0$ and let $\gamma, \sigma : [0, \infty) \to M$ be two geodesics, parametrized by arclength, with $\gamma(0) = \sigma(0)$. We assume that $\gamma$ is a ray and that $\alpha := \sphericalangle(\dot\gamma(0), \dot\sigma(0)) < \frac{\pi}{2}$.
Show that $\lim\limits_{t \to \infty} d(\sigma(0), \sigma(t)) = \infty$.
*Hint: Using the triangle inequality, show first that it is enough to prove:* $\lim\limits_{s \to \infty} \left(d(\gamma(s), \sigma(t)) - d(\gamma(s), \gamma(0))\right) \geq t \cos \alpha$, *for any fixed* $t \geq 0$. *Then apply Toponogov's Theorem (A).*

*Hand in the solutions on* ***Monday, July 15, 2013*** *before the lecture.*