Spectral estimates on 2-tori

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Abstract

We prove upper and lower bounds for the eigenvalues of the Dirac operator and the Laplace operator on 2-dimensional tori. In particluar we give a lower bound for the first eigenvalue of the Dirac operator for non-trivial spin structures. It is the only explicit estimate for eigenvalues of the Dirac operator known so far that uses information about the spin structure.

As a corollary we obtain lower bounds for the Willmore functional of a torus embedded into S^3.

In the final section we compare Dirac spectra for two different spin structures on an arbitrary Riemannian spin manifold.

Keywords: Dirac operator, Laplace operator, spectrum, conformal metrics, two-dimensional torus, spin structures, Willmore functional

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1 Introduction

The Dirac operator is an elliptic differential operator of order one playing an important role both in modern physics and in mathematics. In physics, particles with non-integer spin, so-called fermions, are described by the Dirac equation. Let us assume that the space-time M is stationary, $M = \mathbb{R} \times N$ and that the spatial component N is compact and admits a spin structure. Then stationary fermions have a wave function of the form

$$\Psi(t, x) = e^{iEt}\Psi_0(x) \qquad t \in \mathbb{R},\, x \in N$$

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where Ψ_0 is an eigenspinor of D_N^2, the square of the Dirac operator on N, that belongs to the eigenvalue λ . The energy E and the eigenvalue λ are related via the formula

$$E^2 = \lambda + m^2$$

with m being the rest mass of the particle. Knowing the spectrum therefore means knowing possible energies. The first eigenvalue is of particular interest as it characterizes the energy of the state of lowest energy — the vacuum. On an arbitrary Riemannian manifold, exact calculation of the spectrum is impossible, thus one tries try to find bounds for the eigenvalues.

Bounding eigenvalues of the Dirac operator on a compact Riemannian manifold N is also an important tool in differential geometry and topology. If N is spin and carries a metric whose scalar curvature is greater than or equal to $s_0 > 0$ at every point, then with the help of the Schrödinger-Lichnerowicz formula it is easy to prove that the first eigenvalue λ_1 of D^2 is bounded from below by $s_0/4$. On the other hand Atiyah-Singer index theorem tells us that positivity of the first eigenvalue of D^2 on a compact Riemannian manifold N implies that the $\hat{A}$-genus vanishes. Therefore any compact spin manifold admitting a positive scalar curvature metric has vanishing $\hat{A}$-genus.

Lower bounds for Dirac eigenvalues can also be applied to problems in classical differential geometry. For any immersion $F : N \to \mathbb{R}^n$ of a compact manifold N, Christian Bär [Bär98b] proved

$$\int_N |H|^2 \geq \mu_1 \mathrm{area}(N). \tag{1}$$

Here N carries the induced metric, μ_1 is the first eigenvalue of the square of a twisted Dirac operator and H is the mean curvature vector field of $F(N) \subset \mathbb{R}^n$. If N is the 2-dimensional torus T^2, then the left hand side of (1) is the so-called Willmore functional. The Willmore conjecture states

$$\int_{T^2} |H|^2 \geq 2\pi^2$$

for any immersion $F : T^2 \to \mathbb{R}^n$. This conjecture first appeared in [Wil65] for the case $n = 3$. In the meantime the conjecture has been verified for several classes of immersions, for example for immersions with rotational symmetry [LS84] or for non-injective immersions [LY82]. Nevertheless the conjecture remains open until now. For further information on this conjecture the reader may read the introductions of [Top98b] or [Amm00].

Now assume for simplicity that F is an embedding and $F(T^2) \subset S^3 \subset \mathbb{R}^4$. In this case, the twisting bundle is trivial, and μ_1 is the first eigenvalue of the square of the classical Dirac operator associated to a non-trivial spin structure. Our goal is to use inequality (1) in order to derive lower bounds for the Willmore functional. If the induced metric on T^2 is flat, the spectrum of D has been explicitely calculated [Fri84] and we obtain a lower bound for $\int_{T^2} |H|^2$.

Obtaining lower eigenvalue estimates for non-flat tori is much harder. John Lott [Lot86, Proposition 1] proved the existence of a constant $C_{\text{Lott}} > 0$ depending on the spin-conformal type of the torus such that

$$\mu_1 \text{area} \geq C_{\text{Lott}}. \tag{2}$$

Unfortunately, Lott's article does not give an explicit value and it seems hard to express such a constant C_{Lott} in terms of meaningful geometric data. Lott's estimate uses the L^p-boundedness of zero order pseudo-differential operators and Sobolev embedding theorems, hence corresponding constants are hard to interpret without using explicit coordinates.

The starting point of the author's PhD thesis [Amm98] and of the present article is to find an explicit lower bound for μ_1 that uses information about the spin structure. All explicit lower estimates known before did not use any information about the spin structure.

For general compact Riemannian manifolds the problem of finding such estimates is rather difficult. It is not clear at all what kind of data from the spin structure could be used in order to get an additional term in a lower eigenvalue estimate. Take for example a compact manifold with non-vanishing $\hat{A}$-genus. It has $\mu_1 = 0$ for any spin structure, thus the contribution of the spin structure in the estimate has to vanish.

As the general case is hard to handle, most of the article will specialize to the 2-dimensional torus T^2. By the uniformization theorem any 2-dimensional torus is conformally equivalent to a flat torus. We use this fact in order to control the geometry. An important, but also very technical step for this is the estimate of the oscillation of the conformal factor (Section 9). Although our main goal was to find lower estimates for the Dirac eigenvalues, it turns out that this method gives upper and lower bounds for all eigenvalues both of the Laplace operator and the Dirac operator and for any spin structure. We prove different versions of the estimates. Theorem 2.2 for example states for the first eigenvalue μ_1 of the square of the Dirac operator

$$\mu_1 \text{area} \geq C_{\text{Ammann}} \cdot \kappa \tag{3}$$

where $C_{\text{Ammann}} > 0$ is an explicit constant depending on the spin-conformal class and $\kappa \leq 1$ is a curvature expression that satisfies $\kappa = 1$ if the metric is flat. This estimate is sharp for any flat metric.

In view of Lott's result (2), it is tempting to conjecture that we can drop the curvature term, i. e. $\mu_i \text{area} \geq C_{\text{Ammann}}$. This is false however: we can prove by example at the end of section 12 that for many spin-conformal structures the optimal constant in Lott's estimate is not attained by a flat torus.

In section 12 we will prove some lower bounds for the Willmore functional that are strongly related to our lower estimates of the Dirac eigenvalues. In particular we prove for embeddings $T^2 \to S^3$ that under a curvature condition the Willmore functional converges to ∞ if the spin-conformal type of the embedding converges to one end of the spin-conformal moduli space (Corollary 12.5).

The results in this paper about the Willmore conjecture are strongly related to another preprint of the author [Amm00]. The results of the present article are stronger near one of the ends of the spin-conformal moduli space but they have other drawbacks. Namely, they do not generalize easily to higher codimensions and they impose a restriction on the spin-conformal class.

The structure of the article is as follows: In section 2 we will state our spectral estimates on 2-tori. Sections 3 to 11 provide proofs of the statements in section 2. We then apply Theorem 9.1 once again and derive an application to the Willmore functional that is related to our lower eigenvalue estimates.

Finally in section 13 we will prove a result for arbitrary spin manifolds M. Let M carry two different spin structures ϑ and ϑ'. The difference of these spin structures $\chi := \vartheta - \vartheta'$ is an element in $H^1(M,\mathbb{Z}_2) = \mathrm{Hom}_{\mathbb{Z}}(H_1(M,\mathbb{Z}),\mathbb{Z}_2)$. Assume that χ vanishes on the torsion part of $H_1(M,\mathbb{Z})$. We will define a norm $\|\chi\|_{L^\infty}$, the stable norm of χ. We prove that the eigenvalues $(\rho_i)_{i\in\mathbb{Z}}$ of the Dirac operator corresponding to ϑ and the eigenvalues $(\rho_i')_{i\in\mathbb{Z}}$ corresponding to ϑ' can be numbered so that

$$|\rho_i - \rho_i'| \leq 2\pi \, \|\vartheta - \vartheta'\|_{L^\infty} .$$

If the spectrum is known for ϑ and if $|\rho_i| > 2\pi \, \|\vartheta - \vartheta'\|_{L^\infty}$ for any $i \in \mathbb{Z}$, then this yields a lower bound for any ρ_i'.

At the end of the introduction we want to mention some other publications that treat the interplay between spin structures and the spectrum of the Dirac operator. However, they do not derive explicit eigenvalue bounds for generic metrics. We will restrict to the most recent ones. For further references and a good overview of the subject we refer to [Bär00].

Dahl [Dah99] shows that the difference of the eta-invariants corresponding to two different spin structures is an integer, if the difference of the spin structures viewed as an element in $\mathrm{Hom}_{\mathbb{Z}}(H_1(M,\mathbb{Z}),\mathbb{Z}_2)$ vanishes on the torsion part. Bär [Bär98a] calculated the essential spectrum of hyperbolic 2- and 3-manifolds of finite volume. In these examples, the essential spectrum depends on the spin structure at the cusps. Pfäffle [Pfä99] calculated the spectrum and the η-invariants of flat Bieberbach manifolds. These spectra also depend on the spin structure.

Several results in the present article already appeared in the author's PhD thesis [Amm98].

2 Main results

In this section we summarize our results about the spectra of Dirac and Laplace operators on 2-tori.

The spectrum of the Dirac operator depends on the spin structure. At first, we recall some important facts about spin structures and introduce some notation. Spin structures will be discussed in more detail in section 4.

Let M be a compact orientable manifold with vanishing second Stiefel-Whitney class $w_2(TM) = 0$. Such manifolds admit a spin structure. However, the spin structure is not unique in general. The group $H^1(M, \mathbb{Z}_2)$ acts freely and transitively on the set of spin structures $\mathfrak{Spin}(M)$, i. e. $\mathfrak{Spin}(M)$ is an affine space associated to the vector space $H^1(M, \mathbb{Z}_2)$. After fixing a spin structure and a Riemannian metric on M we can define the spinor bundle $\Sigma M \to M$ and a Dirac operator $D : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$.

We are mainly interested in the case $M = T^2$. The 2-dimensional torus T^2 is spin. Because of $\# \mathfrak{Spin}(T^2) = \# H^1(T^2, \mathbb{Z}_2) = 4$ there are 4 spin structures on T^2. There is exactly one spin structure in $\mathfrak{Spin}(T^2)$ for which 0 lies in the spectrum of D, regardless of the underlying metric g. This spin structure will be called *trivial* (see section 4 for other characterizations). We will identify the trivial spin structure with $0 \in H^1(T^2, \mathbb{Z}_2)$. This identification yields an identification of the affine space $\mathfrak{Spin}(T^2)$ with $H^1(T^2, \mathbb{Z}_2)$. On the other hand, we will identify $H^1(T^2, \mathbb{Z}_2)$ with $\mathrm{Hom}_{\mathbb{Z}}(H_1(T^2, \mathbb{Z}), \mathbb{Z}_2)$. Hence spin structures on T^2 are in a canonical one-to-one relation to such homomorphisms. Frequently, we will use the term "*spin homomorphism*" instead of "spin structure" in order to indicate that we regard the spin structure as an element in $\mathrm{Hom}_{\mathbb{Z}}(H_1(T^2, \mathbb{Z}), \mathbb{Z}_2)$.

If the torus T^2 carries a flat metric, it is very helpful to write the torus as $\mathbb{R}^2/\Gamma$ with a lattice $\Gamma \cong H_1(T^2, \mathbb{Z})$. We always assume that $\mathbb{R}^2/\Gamma$ carries the metric induced by the Euclidean metric on $\mathbb{R}^2$. Let Γ^* be the lattice dual to Γ. Elements $\chi \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z}_2)$ are represented by vectors $\alpha \in (1/2)\Gamma^*$ with the property

$$\chi(x) = (-1)^{2\alpha(x)} \qquad \forall x \in \Gamma.$$

Note that χ determines α only up to elements in Γ^*.

We define the function $\mathcal{S} : [0, 4\pi[\times [0, \infty[\times]1, \infty[\times]0, \infty] \to]0, \infty]$ by

$$\mathcal{S}(\mathcal{K}, \mathcal{K}', p, \mathcal{V}) := \frac{p}{p-1} \left[\frac{\mathcal{K}'}{4\pi} + \frac{1}{2} \left| \log\left(1 - \frac{\mathcal{K}}{4\pi}\right) \right| + \frac{\mathcal{K}}{8\pi - 2\mathcal{K}} \log\left(\frac{2\mathcal{K}'}{\mathcal{K}}\right) \right] + \frac{\mathcal{K}\mathcal{V}}{8}$$

for $\mathcal{K} > 0$ and $\mathcal{S}(0, \mathcal{K}', p, \mathcal{V}) := 0$.

Let Area_g be the area of (T^2, g).

THEOREM 2.1. *Let (T^2, g) be a Riemannian 2-torus with spin homomorphism χ. Choose a lattice Γ in $\mathbb{R}^2$ with $\mathrm{vol}(\mathbb{R}^2/\Gamma) = 1$ together with a conformal map $A : \mathbb{R}^2/\Gamma \to (T^2, g)$.*

Assume that $A^*(\chi)$ is represented by $\alpha \in (1/2)\Gamma^*$. Let $0 \le \ell_0 \le \ell_1 \le \ell_2 \le \dots$ be the sequence of lengths of $\Gamma^* + \alpha$ (with multiplicities), and let $(\mu_i \,|\, i = 1, 2, \dots)$ be the spectrum of D^2 on (T^2, g, χ).

Then

$$e^{-2\,\mathrm{osc}\,u}\, 4\pi^2\, \ell^2_{\left[\frac{i-1}{2}\right]} \le \mu_i \,\mathrm{Area}_g \le e^{2\,\mathrm{osc}\,u}\, 4\pi^2\, \ell^2_{\left[\frac{i-1}{2}\right]}.$$

If $\|K_g\|_{L^1(T^2,g)} < 4\pi$, then

$$\mathrm{osc}\, u \le \mathcal{S}\left(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)} \,\mathrm{Area}_g{}^{1-(1/p)}, p, \sigma_1(T^2,g)^{-2}\right) \tag{4}$$

with $\sigma_1(T^2, g) := \inf\left\{\mathrm{length}(\beta) \,|\, \beta \in \Gamma - \{0\}\right\}$.

The number $\sigma_1(T^2, g)$ is a conformal invariant of (T^2, g) which will be called *cosystole*.

The most difficult step in the proof of this theorem is to find the estimate (4). This step will be performed in Theorem 9.1. For proving the above theorem, we will use the explicit formula for the spectra of flat tori (Proposition 7.2, [Fri84]). Another important tool for the proof is the following proposition.

PROPOSITION 5.2. *Let M be a compact manifold with two conformal metrics $\tilde g$ and $g = e^{2u}\tilde g$. Let D and $\widetilde D$ be the corresponding Dirac operators with respect to a common spin structure. We denote the eigenvalues of D^2 by $\mu_1 \le \mu_2 \le \dots$ and the ones of $\widetilde D^2$ by $\widetilde\mu_1 \le \widetilde\mu_2 \le \dots$.*

Then

$$\mu_i \min_{m\in M} e^{2u(m)} \le \widetilde\mu_i \le \mu_i \max_{m\in M} e^{2u(m)} \qquad \forall i = 1, 2, \dots .$$

This proposition is based on Hitchin's transformation formula for spinors [Hit74] (see section 5 for a proof).

In section 6 we will define a norm on $H^1(T^2, \mathbb{Z}_2)$, the L^2-norm. This norm allows us to derive explicit lower bounds for the first eigenvalue of D^2 on T^2. This lower bound is non-trivial if the spin structure is non-trivial. The cosystole $\sigma_1(T^2, g)$ can also be expressed in terms of the L^2-norm

$$\sigma_1(T^2, g) := \inf\left\{\|\alpha\|_{L^2} \,|\, \alpha \in H^1(T^2, \mathbb{Z}_2), \quad \alpha \neq 0\right\}.$$

(see section 6, in particular Proposition 6.1 (a)).

THEOREM 2.2. *Let (T^2, g) be a Riemannian 2-torus with spin homomorphism χ. Assume that $\|K_g\|_{L^1(T^2,g)} < 4\pi$. Then the first eigenvalue μ_1 of D^2 satisfies*

$$\mu_1 \mathrm{Area}_g \ge \frac{4\pi^2\, \|\chi\|^2_{L^2}}{\exp\left(2\mathcal{S}(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)} \,\mathrm{Area}_g{}^{1-(1/p)}, p, \sigma_1(T^2,g)^{-2}\right)},$$

The equality is attained if and only if g is flat.

From this theorem we will obtain two corollaries estimating μ_1 in terms of the *systole* sys_1, the *spinning systole* spin-sys$_1$ and the *non-spinning systole* nonspin-sys$_1$.

$$\begin{aligned}
\mathrm{sys}_1(T^2,g) &:= \inf\{\text{length}(\gamma)\,|\,\gamma \text{ is a non-contractible loop.}\}\\
\text{spin-sys}_1(T^2,g,\chi) &:= \inf\{\text{length}(\gamma)\,|\,\gamma \text{ is a loop with } \chi([\gamma])=-1.\}\\
\text{nonspin-sys}_1(T^2,g,\chi) &:= \inf\{\text{length}(\gamma)\,|\,\gamma \text{ is a non-contractible loop with } \chi([\gamma])=1\\
&\qquad\qquad \text{and } [\gamma] \text{ is a primitive element in } H_1(T^2,\mathbb{Z}).\}
\end{aligned}$$

An element $\alpha \in H_1(T^2,\mathbb{Z})$ is called *primitive* if there are no $k\in\mathbb{N}$, $k\geq 2$, $\beta\in H_1(T^2,\mathbb{Z})$ with $\alpha = k\cdot\beta$.

COROLLARY 2.3. *Let (T^2,g) be a Riemannian 2-torus with* ***non-trivial*** *spin homomorphism χ. Assume that $\|K_g\|_{L^1(T^2,g)} < 4\pi$. Then the first eigenvalue μ_1 of D^2 satisfies*

$$\mu_1 \mathrm{Area}_g{}^2 \geq \frac{\pi^2\,\text{nonspin-sys}_1(T^2,g,\chi)^2}{\exp\Big(2\mathcal{S}(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)}\,\mathrm{Area}_g{}^{1-(1/p)}, p, \frac{\mathrm{Area}_g}{\mathrm{sys}_1(T^2,g)^2}\Big)}.$$

The equality is attained if and only if g is flat.

COROLLARY 2.4. *Let (T^2,g) be a Riemannian 2-torus with* ***non-trivial*** *spin homomorphism χ. Assume that $\|K_g\|_{L^1(T^2,g)} < 4\pi$. Then the first eigenvalue μ_1 of D^2 satisfies*

$$\mu_1 \text{spin-sys}_1(T^2,g,\chi)^2 \geq \frac{\pi^2}{\exp\Big(4\mathcal{S}(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)}\,\mathrm{Area}_g{}^{1-(1/p)}, p, \frac{\mathrm{Area}_g}{\mathrm{sys}_1(T^2,g)^2}\Big)}.$$

The equality is attained if and only if

(a) g is flat, i. e. (T^2,g) is isometric to $\mathbb{R}^2/\Gamma$ for a suitable lattice Γ, and

(b) there are generators γ_1,γ_2 for Γ statisfying $\gamma_1\perp\gamma_2$, $\chi(\gamma_1)=1$ and $\chi(\gamma_2)=-1$.

Using Proposition 6.1 and the inequalities from section 10 the two corollaries immediately follow from Theorem 2.2.

We now turn to the Laplace operator and to the Dirac operator associated to a trivial spin structure. We recall a well-known proposition that is the analogue of Proposition 5.2 for the Laplacian on surfaces (section 5).

PROPOSITION 5.1. *Let M be a compact 2-dimensional manifold with two conformal metrics $\tilde g$ and $g = e^{2u}\tilde g$. The eigenvalues of the Laplacian on functions corresponding to g and $\tilde g$ will be denoted as $0=\lambda_0<\lambda_1\leq\lambda_2\dots$ and $0=\tilde\lambda_0<\tilde\lambda_1\leq\tilde\lambda_2\dots$ respectively.*

Then
$$\lambda_i \min_{m\in M} e^{2u(m)} \leq \tilde\lambda_i \leq \lambda_i \max_{m\in M} e^{2u(m)} \qquad \forall i = 1,2,\dots.$$

Together with Proposition 7.1 and Theorem 9.1 we obtain

THEOREM 2.5. *Let (T^2, g) be a torus conformally equivalent to $\mathbb{R}^2/\Gamma$, $\mathrm{vol}(\mathbb{R}^2/\Gamma) = 1$. Let Γ^* be the lattice dual to Γ. Let $0 \le \ell_0 \le \ell_1 \le \ell_2 \le \ldots$ be the sequence of lengths of Γ^*, and let $(\lambda_i \,|\, i = 0, 1, 2, \ldots)$ be the spectrum of the Laplacian on functions on (T^2, g), then*

$$e^{-2\,\mathrm{osc}\,u}\, 4\pi^2\, \ell_i^2 \le \lambda_i\, \mathrm{Area}_g \le e^{2\,\mathrm{osc}\,u}\, 4\pi^2\, \ell_i^2.$$

If $\|K_g\|_{L^1(T^2,g)} < 4\pi$, then

$$\mathrm{osc}\, u \le \mathcal{S}\left(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)} \,{\mathrm{Area}_g}^{1-(1/p)}, p, \sigma_1(T^2,g)^{-2}\right).$$

Note that this theorem also provides bounds for the Laplacian on forms: By Poincaré duality the spectrum on 2-forms is the same as the spectrum on functions, and the Laplacian on 1-forms also has the same non-zero eigenvalues, but each with multiplicity two.

The theorem implies, in particular, a lower bound on the first positive eigenvalue.

THEOREM 2.6. *Let (T^2, g) be a Riemannian 2-torus. Assume that $\|K_g\|_{L^1(T^2,g)} < 4\pi$. Then the first positive eigenvalue λ_1 of the Laplacian on functions satisfies*

$$\lambda_1 \mathrm{Area}_g \ge \frac{4\,\pi^2\,\sigma_1(T^2,g)^2}{\exp\left(2\mathcal{S}(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)} \,{\mathrm{Area}_g}^{1-(1/p)}, p, \sigma_1(T^2,g)^{-2}\right)}.$$

The equality is attained if and only if g is flat.

COROLLARY 2.7. *Let (T^2, g) be a Riemannian 2-torus. Assume that $\|K_g\|_{L^1(T^2,g)} < 4\pi$. Then the first positive eigenvalue λ_1 of the Laplacian on functions satisfies*

$$\lambda_1 {\mathrm{Area}_g}^2 \ge \frac{4\,\pi^2\,\mathrm{sys}_1(T^2,g)^2}{\exp\left(2\mathcal{S}(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)} \,{\mathrm{Area}_g}^{1-(1/p)}, p, \frac{\mathrm{Area}_g}{\mathrm{sys}_1(T^2,g)^2}\right)}.$$

The equality is attained if and only if g is flat.

Remark. Theorem 2.6 and Corollary 2.7 also hold for the first positve eigenvalue of D^2, if the spin structure is trivial. Theorem 2.5 holds for the spectrum of D^2, if we double the multiplicities.

The structure of the paper is as follows: In the following sections (sections 3–11) we will prove our main results. In section 12, we will apply the inequalities in Proposition 9.1 in order to obtain a lower bound on the Willmore functional. Finally, in section 13 we assume that a manifold of abitrary dimension $n \ge 2$ carries two spin structures. We derive an upper bound for the spectra of the corresponding Dirac operators.

3 Overview

We want to obtain upper and lower bounds for the eigenvalues of the Dirac operator and the Laplace operator on a Riemannian 2-torus (T^2, g).

The Clifford action of the volume element on spinors anticommutes with the Dirac operator D. Thus, the spectrum of D is symmetric and is uniquely determined by the spectrum of its square D^2. Therefore we will study the spectrum of D^2 instead of the spectrum of D. In the literature D^2 is often called the Dirac Laplacian.

In order to prove bounds on eigenvalues we use the uniformization theorem which tells us that we can write g as $g = e^{2u} g_0$ with a real-valued function u and a flat metric g_0. For flat tori the spectrum of the Laplacian and the Dirac operator is known: the spectra can be calculated in terms of the dual lattice corresponding to (T^2, g_0).

We obtain bounds through the following steps.

(a) Comparison of the spectrum of (T^2, g) and the spectrum of (T^2, g_0) (Propositions 5.1 and 5.2).

(b) Introduction of certain spin-conformal invariants that contain information about the dual lattice corresponding to (T^2, g_0) (section 6).

(c) The knowledge of spectra of flat tori (section 7).

(d) A bound on $\operatorname{osc} u = \max u - \min u$ (section 9).

(e) Derivation, in section 10, of certain inequalities that are in a sense inverse to the inequalities in Proposition 6.1 and contain a curvature term.

In section 11, we combine the inequalities and derive the main results.

4 Spin structures

The eigenvalues of D depend on the spin structures and we want to find estimates depending on the spin structure. In this section we recall some important facts about spin structures. Good references about spin structures are [LM89], [BG92] and [Swi93, section II]. We will define spin structures without fixing a Riemannian metric. This definition will allow us to identify spin structures on diffeomorphic but not isometric manifolds (see Proposition 5.2).

Let M be an oriented manifold of dimension $n \geq 2$. The bundle $\mathrm{GL}^+(M)$ of oriented bases over M is a principal $\mathrm{GL}^+(n, \mathbb{R})$-bundle. The fundamental group of $\mathrm{GL}^+(n, \mathbb{R})$ is

$\mathbb{Z}$ for $n = 2$ and $\mathbb{Z}_2$ for $n \geq 3$. Therefore $\mathrm{GL}^+(n,\mathbb{R})$ has a unique connected double covering $\Theta : \widetilde{\mathrm{GL}^+}(n,\mathbb{R}) \to \mathrm{GL}^+(n,\mathbb{R})$.

Definition. A spin structure on M is a pair $(\widetilde{\mathrm{GL}^+}(M), \vartheta)$ where $\widetilde{\mathrm{GL}^+}(M)$ is a principal $\widetilde{\mathrm{GL}^+}(n,\mathbb{R})$-bundle over M and ϑ is a double covering $\widetilde{\mathrm{GL}^+}(M) \to \mathrm{GL}^+(M)$ such that

$$\begin{array}{ccccc} \widetilde{\mathrm{GL}^+}(M) \times \widetilde{\mathrm{GL}^+}(n,\mathbb{R}) & \to & \widetilde{\mathrm{GL}^+}(M) & & \\ & & & \searrow & \\ \downarrow \vartheta \times \Theta & & \downarrow \vartheta & & M \\ & & & \nearrow & \\ \mathrm{GL}^+(M) \times \mathrm{GL}^+(n,\mathbb{R}) & \to & \mathrm{GL}^+(M) & & \end{array} \tag{5}$$

commutes. The horizontal arrows are given by the group action.

There is a spin structure on M if and only if the second Stiefel-Whitney class $w_2(TM)$ vanishes. Such manifolds are called *spin*. From now on we assume that M is spin.

Two spin structures $(\widetilde{\mathrm{GL}^+}(M), \vartheta)$ and $(\widetilde{\mathrm{GL}}_1^+(M), \vartheta_1)$ are identified if there is a fiber preserving isomorphism of principal $\widetilde{\mathrm{GL}^+}(n,\mathbb{R})$-bundles $\alpha : \widetilde{\mathrm{GL}^+}(M) \to \widetilde{\mathrm{GL}}_1^+(M)$ with $\vartheta = \vartheta_1 \circ \alpha$.

The set of all spin structures $(\widetilde{\mathrm{GL}^+}(M), \vartheta)$ over M will be denoted by $\mathfrak{Spin}(M)$. The set $\mathfrak{Spin}(M)$ has the structure of an affine space associated to the vector space $H^1(M,\mathbb{Z}_2)$, i. e. $H^1(M,\mathbb{Z}_2)$ acts freely and transitively on $\mathfrak{Spin}(M)$. We will describe this action: Elements in $H^1(M,\mathbb{Z}_2)$ can be viewed as principal $\mathbb{Z}_2$-bundles over M [LM89, Appendix A]. Let $\pi : P_\chi \to M$ be the $\mathbb{Z}_2$-bundle defined by $\chi \in H^1(M,\mathbb{Z}_2)$. Let $(\widetilde{\mathrm{GL}^+}(M), \vartheta)$ be a spin structure. The group $\mathbb{Z}_2$ acts by deck-transformation both on $\widetilde{\mathrm{GL}^+}(M)$ and P_χ. We define

$$\widetilde{\mathrm{GL}}_1^+(M) := (\widetilde{\mathrm{GL}^+}(M) \times_M P_\chi)/\mathbb{Z}_2$$

where $\mathbb{Z}_2$ acts diagonally on the fiberwise product of the bundles. The map

$$\vartheta \times_M \pi : \widetilde{\mathrm{GL}^+}(M) \times_M P_\chi \to \mathrm{GL}^+(M) \qquad (A, \alpha) \mapsto \vartheta(A)$$

is invariant under the $\mathbb{Z}_2$-action and therefore defines a map $\vartheta_1 : \widetilde{\mathrm{GL}}_1^+(M) \to \mathrm{GL}^+(M)$ compatible with (5). The action of χ maps $(\widetilde{\mathrm{GL}^+}(M), \vartheta)$ to the spin structure $(\widetilde{\mathrm{GL}}_1^+(M), \vartheta_1)$. This action is free and transitive [LM89, II§1].

Now we fix a Riemannian metric g on M. This reduces our structure group from $\mathrm{GL}^+(n,\mathbb{R})$ to $\mathrm{SO}(n)$. The bundle of positively oriented orthonormal bases $\mathrm{SO}(M,g)$ is a principal $SO(n)$-bundle. The *spin group* is defined by $\mathrm{Spin}(n) := \Theta^{-1}(\mathrm{SO}(n))$ and is the unique connected double covering of $\mathrm{SO}(n)$. A *metric spin structure* is a pair $(\mathrm{Spin}(M,g), \vartheta)$

where $\mathrm{Spin}(M,g)$ is a principal $\mathrm{Spin}(n)$-bundle over M and ϑ is a double covering $\mathrm{Spin}(M,g) \to \mathrm{SO}(M,g)$ satisfying a compatibility condition analogous to (5). For any spin structure $(\widetilde{\mathrm{GL}^+}(M), \vartheta)$ we obtain a metric spin structure $(\mathrm{Spin}(M,g), \vartheta')$ by restriction:

$$\mathrm{Spin}(M,g) := \vartheta^{-1}(\mathrm{SO}(M,g)) \qquad \vartheta' := \vartheta|_{\mathrm{Spin}(M,g)}.$$

Via this restriction map, the set of metric spin structures is in a natural one-to-one correspondance to $\mathfrak{Spin}(M)$ [Swi93].

Metric spin structures are used to define spinors and the Dirac operator. Let $\gamma_n : \mathrm{Spin}(n) \to \mathrm{SU}(\Sigma_n)$ be the complex spinor representation of $\mathrm{Spin}(n)$. This is a complex representation of dimension $2^{[n/2]}$. It is irreducible for n odd. For n even, it consists of two irreducible components γ_n^+ and γ_n^-, $\gamma_n^\pm : \mathrm{Spin}(n) \to \mathrm{SU}(\Sigma_n^\pm)$. The representation γ_n is not a pullback from a representation of $\mathrm{SO}(n)$. The associated vector bundle $\Sigma M := \mathrm{Spin}(M) \times_{\gamma_n} \Sigma_n$ is called *spinor bundle* and its sections are *spinors*. The Dirac operator (see [LM89] for a definition) is an elliptic operator acting on the space of smooth spinors.

Large parts of this article will deal with the case $M = T^2$. In this case many of our definitions simplify. Let $f : \mathbb{R}^2 \to T^2$ be a smooth covering map with deck transformation group $\mathbb{Z}^2$ acting by translation. Then

$$\begin{array}{rcl} \tau_f : T^2 \times \mathrm{GL}^+(2) & \to & \mathrm{GL}^+(T^2) \\ (f(p), A) & \mapsto & (\partial_x f(p), \partial_y f(p)) \cdot A \end{array}$$

yields a trivialization of $\mathrm{GL}^+(T^2)$.

Definition. The *trivial spin structure on T^2* (with respect to f) is the one given by $\sigma_f := (\widetilde{\mathrm{GL}^+}(T^2), \vartheta)$ with

$$\widetilde{\mathrm{GL}^+}(T^2) := T^2 \times \widetilde{\mathrm{GL}^+}(2) \qquad \vartheta := \tau_f \circ (\mathrm{id} \times \Theta).$$

Consider the bijection

$$\iota_f : H^1(T^2, \mathbb{Z}_2) \to \mathfrak{Spin}(T^2), \quad \chi \mapsto \chi + \sigma_f.$$

The following proposition shows that ι_f does not depend on the choice of f. This will allow us to identify $H^1(T^2, \mathbb{Z}_2)$ and $\mathfrak{Spin}(T^2)$ via ι_f.

PROPOSITION 4.1. *Let $(\widetilde{\mathrm{GL}^+}(T^2), \vartheta)$ be a spin structure on T^2. Let χ_f be the element in $H^1(T^2, \mathbb{Z}_2) = \mathrm{Hom}_{\mathbb{Z}}(H_1(M, \mathbb{Z}), \mathbb{Z}_2)$ with $\iota_f(\chi_f) = (\widetilde{\mathrm{GL}^+}(T^2), \vartheta)$. Fix a complex structure J on TT^2.*

Then for any non-contractible smooth embedding $c : S^1 \to T^2$ the following conditions are equivalent

(1) $\chi_f([c]) = 1$.

(2) $(\dot{c}, J(\dot{c})) : S^1 \to \mathrm{GL}^+(T^2)$ *lifts to* $\widetilde{\mathrm{GL}^+}(T^2)$ *via* ϑ.

Characterization (2) is independent from the choice of f, characterization (1) is independent from the choice of J. Therefore ι_f depends neither on f nor J. The above proposition is an immediate consequence of the following lemma.

LEMMA 4.2. *Let* $c : S^1 \to \mathbb{R}^2/\mathbb{Z}^2$ *be a non-contractible smooth embedding. Choose a lift* $C : \mathbb{R} \to \mathbb{R}^2 = \mathbb{C}$*, i. e.* $C(t)/\mathbb{Z}^2 = c(e^{2\pi it})$ *. Then*

(1) the homology class $[c] \in H_1(\mathbb{R}^2/\mathbb{Z}^2, \mathbb{Z})$ *is primitive, i. e. not a multiple of another element in* $H_1(\mathbb{R}^2/\mathbb{Z}^2, \mathbb{Z})$*.*

(2) the map

$$v(c) : S^1 \to S^1, e^{2\pi it} \mapsto \frac{\dot{C}(t)}{|\dot{C}(t)|}$$

has degree 0*.*

Proof. The curve c can be lifted to the cylinder $Z := \mathbb{R}^2/\langle[c]\rangle$. The lift will be denoted by c^Z. It is a simple closed curve generating $\pi_1(Z)$. By Jordan's theorem about simple closed curves in $\mathbb{R}^2$ we know that this curve divides Z into two connected components Z^+ and Z^-. Each of the components contains one end of the cylinder.

Let us assume that $[c]$ is not primitive, i. e. $[c] = k \cdot a$ with $k \in \mathbb{N}$, $k \geq 2$ and $a \in H_1(M, \mathbb{Z})$ primitive. The action of a on Z maps Z^+ to Z^+ and Z^- to Z^-. Hence the image of c^Z is mapped to itself. This contradicts $k \geq 2$. Thus we have proven (1).

Now let $c_1 : S^1 \to T^2$ be another embedding, homotopic to c. A suitable lift c_1^Z of c_1 divides Z^+ into a bounded and an unbounded part. The bounded part has c^Z and c_1^Z as boundaries and has Euler characteristic 0. Therefore the Gauss-Bonnet theorem for the Euclidean metric on Z yields $v(c) = v(c_1)$. Thus the lemma only has to be checked for one representative in each primitive class. As this is trivial, (2) follows. $\square$

From now on we will identify $\mathfrak{Spin}(T^2)$ with $H^1(T^2, \mathbb{Z}_2)$ and $\mathrm{Hom}_{\mathbb{Z}}(H_1(T^2, \mathbb{Z}), \mathbb{Z}_2)$. Frequently, we will use the term "*spin homomorphism*" instead of "spin structure" in order to indicate that we regard the spin structure as an element in $\mathrm{Hom}_{\mathbb{Z}}(H_1(T^2, \mathbb{Z}), \mathbb{Z}_2)$.

From Proposition 5.2 below it is clear that the trivial spin structure is the only spin structure such that 0 is in the spectrum of the Dirac operator D. Therefore our definition of "trivial spin structure" coincides with the definition in section 2.

Remark. On oriented surfaces there is an alternative approach to define spin structures. We fix a conformal structure on M. Therefore TM is complex line bundle. A *line bundle spin structure* is a pair $(\Sigma^+ M, \vartheta)$ of a complex line bundle $\Sigma^+ M$ and a map $\vartheta : \Sigma^+ M \to TM$ satisfying

$$\vartheta(z \cdot q) = z^2 \cdot \vartheta(q), \qquad \forall q \in \Sigma^+ M, \quad z \in \mathbb{C}.$$

It is not hard to show that there is a natural bijection from the set of line bundle spin structures to the set of spin structures. For $M = T^2$, the trivial spin structure is characterized by the fact that for any non-contractible embedding $S^1 \to T^2$ the tangent vector field $\dot{c} : S^1 \to TT^2$ lifts to $\Sigma^+ T^2$. The line bundle spin structure definition is used by [KS97] for example. The Arf invariant [KS97] can also be used to distinguish the trivial spin structure from the non-trivial ones. The Arf invariant is equal to -1 for the trivial spin structure, and equal to 1 for all others.

5 Comparing spectra of conformal manifolds

In this section we will compare Dirac and Laplace eigenvalues on 2-tori. We recall a proof of a well-known proposition (see e.g. [Dod82, Proposition 3.3] for a more general version).

PROPOSITION 5.1. *Let M be a compact 2-dimensional manifold with two conformal metrics $\tilde{g}$ and $g = e^{2u}\tilde{g}$. The eigenvalues of the Laplacian on functions corresponding to g and $\tilde{g}$ will be denoted as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \dots$ and $0 = \tilde{\lambda}_0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \dots$ respectively.*

Then

$$\lambda_i \min_{m \in M} e^{2u(m)} \leq \widetilde{\lambda}_i \leq \lambda_i \max_{m \in M} e^{2u(m)} \qquad \forall i = 1, 2, \dots.$$

Proof. Let $f_0, \dots, f_i$ be eigenfunctions of Δ_g to the eigenvalues $\lambda_0, \dots, \lambda_i$. Let U_i be the subspace of $V := C^\infty(T^2)$ generated by $f_0, \dots, f_i$. We are bounding $\widetilde{\lambda}_i$ by the Rayleigh quotient:

$$\widetilde{\lambda}_i \leq \max_{f \in U_i - \{0\}} \frac{(\Delta_{\tilde{g}} f, f)_{\tilde{g}}}{(f, f)_{\tilde{g}}}.$$

We obtain for the numerator and the denominator:

$$\begin{aligned}(\Delta_{\tilde{g}} f, f)_{\tilde{g}} &= \int (\Delta_{\tilde{g}} f)\bar{f} \,\mathrm{dvol}_{\tilde{g}} = \int (\Delta_g f)\bar{f} \,\mathrm{dvol}_g \\ &= (\Delta_g f, f)_g \leq \lambda_i (f, f)_g\end{aligned}$$

$$(f, f)_{\tilde{g}} = \int f\bar{f} \,\mathrm{dvol}_{\tilde{g}} = \int f\bar{f}\, e^{-2u} \,\mathrm{dvol}_g \geq e^{-2\max u}\,(f, f)_g.$$

Therefore we obtain

$$\widetilde{\lambda}_i \leq \lambda_i \, e^{2\max u}.$$

The other inequality can be proven in a completely analogous way. □

There is a similar proposition for the Dirac operator.

PROPOSITION 5.2. *Let M be a compact manifold with two conformal metrics $\tilde{g}$ and $g = e^{2u}\tilde{g}$. Let D and $\widetilde{D}$ be the corresponding Dirac operators with respect to a common spin structure. We denote the eigenvalues of D^2 by $\mu_1 \leq \mu_2 \leq \ldots$ and the ones of $\widetilde{D}^2$ by $\widetilde{\mu}_1 \leq \widetilde{\mu}_2 \leq \ldots$.*

Then

$$\mu_i \min_{m\in M} e^{2u(m)} \leq \widetilde{\mu}_i \leq \mu_i \max_{m\in M} e^{2u(m)} \qquad \forall i = 1, 2, \ldots$$

Proof. Let $n := \dim M$. We have

$$\mathrm{dvol}_g = e^{nu}\mathrm{dvol}_{\tilde{g}}.$$

There is an isomorphism of vector bundles [Hit74], [Bau81, Satz 3.14] or [Hij86, 4.3.1]

$$\begin{array}{ccc} \Sigma M & \to & \widetilde{\Sigma} M \\ \Psi & \mapsto & \widetilde{\Psi} \end{array}$$

over the identity $\mathrm{id} : M \to M$ satisfying

$$\widetilde{D}(\widetilde{\Psi}) = e^u \widetilde{D\Psi}$$

and

$$|\widetilde{\Psi}| = e^{\frac{n-1}{2}u}|\Psi|.$$

Let $(\Psi_i \,|\, i = 1, 2, \ldots)$ be an orthonormal basis of the sections of ΣM with Ψ_i being an eigenspinor of D^2 to the eigenvalue μ_i. The vector space spanned by $\Psi_1, \ldots, \Psi_i$ will be denoted by U_i.

We can bound $\widetilde{\mu}_i$ by the Rayleigh quotient

$$\widetilde{\mu}_i \leq \max_{\widetilde{\Psi}\in U_i - \{0\}} \frac{(\widetilde{D}\widetilde{\Psi}, \widetilde{D}\widetilde{\Psi})_{\tilde{g}}}{(\widetilde{\Psi}, \widetilde{\Psi})_{\tilde{g}}}.$$

We look at the numerator and the denominator separately:

$$\begin{aligned} (\widetilde{D}\widetilde{\Psi}, \widetilde{D}\widetilde{\Psi})_{\tilde{g}} &= \int e^{2u}\langle \widetilde{D\Psi}, \widetilde{D\Psi}\rangle \,\mathrm{dvol}_{\tilde{g}} \\ &= \int e^{2u+(n-1)u}\langle D\Psi, D\Psi\rangle \,\mathrm{dvol}_{\tilde{g}} \\ &= \int e^{u}\langle D\Psi, D\Psi\rangle \,\mathrm{dvol}_{g} \\ &\leq (D\Psi, D\Psi)_g \max_{m\in M} e^u \\ &\leq \mu_i \,(\Psi, \Psi)_g \max_{m\in M} e^u \end{aligned}$$

$$
\begin{aligned}
(\widetilde{\Psi}, \widetilde{\Psi})_{\tilde{g}} &= \int \langle \widetilde{\Psi}, \widetilde{\Psi} \rangle \, \mathrm{dvol}_{\tilde{g}} \\
&= \int e^{-u} \, \langle \Psi, \Psi \rangle \, \mathrm{dvol}_g \\
&\geq e^{-\max u} (\Psi, \Psi)_g
\end{aligned}
$$

Thus

$$\widetilde{\mu}_i \leq \mu_i \max_{m \in M} e^{2u}$$

which is one of the inequalities stated in the proposition.

The other inequality can be proven in a completely analogous way. □

6 Systoles and norms on $H^1(T^2, \mathbb{Z}_2)$

In this section we define norms on the space of spin structures $\mathfrak{Spin}(M)$. These norms are strongly related to systoles.

Recall that for any compact Riemannian manifold (M, g), the space $H^1(M, \mathbb{R})$ carries a natural L^p-norm defined to be the quotient norm of the L^p-norm on 1-forms

$$\|\alpha\|_{L^p} := \inf \left\{ \|\omega\|_{L^p} \mid \omega \text{ closed 1-form representing } \alpha \right\}.$$

For $p = \infty$ this norm is the so-called stable norm and for $p = \dim M$ it is invariant under conformal changes of the metric.

In our special case $M = T^2$, we know that $\Gamma^* = H^1(T^2, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(H_1(T^2, \mathbb{Z}), \mathbb{Z})$ is a lattice in $H^1(T^2, \mathbb{R})$ and that the surjective map

$$
\begin{aligned}
P : \frac{1}{2}\Gamma^* &\rightarrow \mathrm{Hom}_{\mathbb{Z}}(H_1(T^2, \mathbb{Z}), \mathbb{Z}_2) = H^1(T^2, \mathbb{Z}_2) \\
\alpha(\,.\,) &\mapsto (-1)^{2\alpha(\,.\,)}
\end{aligned}
$$

has kernel Γ^*.

Definition. The L^p-norm on $H^1(T^2, \mathbb{Z}_2)$ is the quotient norm of the L^p-norm on Γ^* with respect to the quotient map P, i. e. for $\eta \in \mathrm{Hom}_{\mathbb{Z}}(H_1(T^2, \mathbb{Z}), \mathbb{Z}_2)$

$$\|\eta\|_{L^p} := \inf \left\{ \|\alpha\|_{L^p} \mid \alpha \in \frac{1}{2}\Gamma^*, \quad P(\alpha) = \eta \right\}.$$

Therefore we have norms on the space of spin structures on T^2. The L^2-norm is of particular interest as it is invariant under conformal changes and therefore it is a spin-conformal invariant. In the following section it will turn out that the smallest eigenvalue of D^2 on a flat torus with spin structure χ is

$$\frac{4\pi^2\,\|\chi\|_{L^2}^2}{\mathrm{area}}.$$

Another quantity will be used for our estimate of $\operatorname{osc} u$ (section 9): The *cosystole* σ_1 is defined to be

$$\sigma_1(T^2,g) := \inf\left\{\|\alpha\|_{L^2}\ |\ \alpha \in \Gamma^* - \{0\}\right\}.$$

For flat tori the first positive eigenvalue of the Laplacian is

$$\frac{4\pi^2\,\sigma_1^2}{\mathrm{area}}.$$

The aim of the rest of this section is to relate the L^2-norms to some systolic data.

Definition. For a Riemannian 2-torus (T^2,g) with spin structure χ we define the *systole* $\mathrm{sys}_1(T^2,g) \in \mathbb{R}$, the *spinning systole* spin-sys$_1(T^2,g,\chi) \in \mathbb{R}\cup\{\infty\}$ and the *non-spinning systole* nonspin-sys$_1(T^2,g,\chi) \in \mathbb{R}$ to be

$$\begin{aligned}
\mathrm{sys}_1(T^2,g) &:= \inf\{\text{length}(\gamma)\,|\,\gamma \text{ is a non-contractible loop.}\}\\
\text{spin-sys}_1(T^2,g,\chi) &:= \inf\{\text{length}(\gamma)\,|\,\gamma \text{ is a loop with } \chi([\gamma]) = -1.\}\\
\text{nonspin-sys}_1(T^2,g,\chi) &:= \inf\{\text{length}(\gamma)\,|\,\gamma \text{ is a non-contractible loop with } \chi([\gamma]) = 1\\
&\qquad\qquad \text{and } [\gamma] \text{ is a primitive element in } H_1(T^2,\mathbb{Z}).\}
\end{aligned}$$

An element $\alpha \in H_1(T^2,\mathbb{Z})$ is called *primitive* if there are no $k \in \mathbb{N}$, $k \geq 2$, $\beta \in H_1(T^2,\mathbb{Z})$ with $\alpha = k\cdot\beta$.

These quantities have the following relationships

$$\mathrm{sys}_1(T^2,g) = \min\{\text{spin-sys}_1(T^2,g,\chi), \text{nonspin-sys}_1(T^2,g,\chi)\}$$

$$\mathrm{sys}_1(\widehat{T^2},g) = \min\{2\cdot\text{spin-sys}_1(T^2,g,\chi), \text{nonspin-sys}_1(T^2,g,\chi)\}$$

where $\widehat{T^2}$ is the covering of T^2 associated to $\ker\chi \subset \pi_1(T^2)$. This covering is 2-fold for non-trivial χ, and $\widehat{T^2} = T^2$ for $\chi \equiv 1$.

PROPOSITION 6.1. *Let g be any Riemannian metric on T^2 and let χ be any spin homomorphism. There is a flat metric g_0 which is conformal to g. This metric g_0 is unique up to a multiplicative constant.*

Furthermore, the following inequalities hold:

(a) $$\frac{\mathrm{sys}_1(T^2,g)^2}{\mathrm{area}(T^2,g)} \leq \frac{\mathrm{sys}_1(T^2,g_0)^2}{\mathrm{area}(T^2,g_0)} = \sigma_1(T^2,g_0)^2 = \sigma_1(T^2,g)^2$$

(b)
$$\frac{\text{nonspin-sys}_1(T^2,g,\chi)^2}{\mathrm{area}(T^2,g)} \le \frac{\text{nonspin-sys}_1(T^2,g_0,\chi)^2}{\mathrm{area}(T^2,g_0)}$$
$$= 4\,\|\chi\|^2_{L^2(T^2,g_0)} = 4\,\|\chi\|^2_{L^2(T^2,g)}$$

(c)
$$\frac{\text{spin-sys}_1(T^2,g,\chi)^2}{\mathrm{area}(T^2,g)} \le \frac{\text{spin-sys}_1(T^2,g_0,\chi)^2}{\mathrm{area}(T^2,g_0)}$$

(d)
$$\frac{\text{spin-sys}_1(T^2,g_0,\chi)^2}{\mathrm{area}(T^2,g_0)} \ge \frac{1}{4\,\|\chi\|^2_{L^2(T^2,g_0)}}$$

(e) *For any* $\eta \in H^1(T^2,\mathbb{Z}_2)$ *and* $1 \le p \le q \le \infty$
$$\begin{aligned}
\|\eta\|_{L^p(T^2,g_0)}\,\mathrm{area}(T^2,g_0)^{-(1/p)} &= \|\eta\|_{L^q(T^2,g_0)}\,\mathrm{area}(T^2,g_0)^{-(1/q)}\\
\|\eta\|_{L^p(T^2,g)}\,\mathrm{area}(T^2,g)^{-(1/p)} &\le \|\eta\|_{L^q(T^2,g)}\,\mathrm{area}(T^2,g)^{-(1/q)}\\
\|\eta\|_{L^2(T^2,g_0)} &= \|\eta\|_{L^2(T^2,g)}
\end{aligned}$$

(f) *For any* $\eta \in H^1(T^2,\mathbb{Z}_2)$ *and* $1 \le p \le 2 \le q \le \infty$
$$\begin{aligned}
\|\eta\|_{L^p(T^2,g)}\,\mathrm{area}(T^2,g)^{\left(\frac{1}{2}-\frac{1}{p}\right)} &\le \|\eta\|_{L^p(T^2,g_0)}\,\mathrm{area}(T^2,g_0)^{\left(\frac{1}{2}-\frac{1}{p}\right)}\\
\|\eta\|_{L^q(T^2,g)}\,\mathrm{area}(T^2,g)^{\left(\frac{1}{2}-\frac{1}{q}\right)} &\ge \|\eta\|_{L^q(T^2,g_0)}\,\mathrm{area}(T^2,g_0)^{\left(\frac{1}{2}-\frac{1}{q}\right)}
\end{aligned}$$

We have equality in the inequalities of (a)–(c) if and only if g is flat.

For the characterization of the equality case in (d) we choose a lattice Γ together with an isometry $I : \mathbb{R}^2/\Gamma \to (T^2,g_0)$. Then equality in (d) is equivalent to the fact that there are generators γ_1, γ_2 for the lattice Γ satisfying $\gamma_1 \perp \gamma_2$, $I^*(\chi)(\gamma_1) = 1$ and $I^*(\chi)(\gamma_2) = -1$.

Proof. The existence and uniqueness of g_0 follows from the uniformization theorem for 2-dimensional tori. The equations for the flat metric g_0 follow directly from elementary calculations. As already stated previously, the L^2-norm is invariant under conformal changes, thus the last equations in (a), (b) and (e) hold. The inequality in (e) follows from the Hölder inequality.

The first equation in (e) then follows from the fact, that η is represented by a real harmonic 1-form ω with $\|\eta\|_{L^1(T^2,g_0)} = \|\omega\|_{L^1(T^2,g_0)}$. The pointwise norm $|\omega|_{g_0}$ is constant and therefore
$$\|\omega\|_{L^1(T^2,g_0)} = \|\omega\|_{L^\infty(T^2,g_0)}\,\mathrm{area}(T^2,g_0) \ge \|\eta\|_{L^\infty(T^2,g_0)}\,\mathrm{area}(T^2,g_0).$$
The inequalities in (f) follow from (e).

The remaining inequalities in (a), (b) and (c) are direct consequences from Lemma 6.2 below.

The discussion of the equality case is straightforward. $\square$

LEMMA 6.2 ([Amm98, Prop. 3.7.2]). *With the notation of the previous proposition we define for $v \in H_1(T^2, \mathbb{Z})$*

$$\mathcal{L}_g(v) := \min \left\{ \text{length}_g(c) \;\middle|\; c : S^1 \to \mathbb{T}^2 \text{ represents } v \right\},$$

and $\mathcal{L}_{g_0}(v)$ similarly. Then

$$\frac{\mathcal{L}_g(v)^2}{\operatorname{area}(T^2, g)} \leq \frac{\mathcal{L}_{g_0}(v)^2}{\operatorname{area}(T^2, g_0)}.$$

We have equality for $v \neq 0$ if and only if g is flat.

Proof of lemma. The proof of lemma follows the pattern of the proof of Loewner's theorem in [Gro81, 4.1].

Let $g = e^{2u} g_0$. We start with a minimizer c of $\mathcal{L}_{g_0}(v)$. There is an isometric torus action on (T^2, g_0) acting by translations. Translation by $x \in T^2$ will be denoted by L_x. An easy calculation shows that

$$\int_{T^2, g_0} dx \, \text{length}_g(L_x(c)) = \mathcal{L}_{g_0}(v) \int_{T^2, g_0} dx \, e^{u(x)} \leq \mathcal{L}_{g_0}(v) \operatorname{area}(T^2, g_0)^{1/2} \operatorname{area}(T^2, g)^{1/2}.$$

Because the left hand side is an upper bound for $\mathcal{L}_g(v) \operatorname{area}(T^2, g_0)$ the inequality of the lemma follows. The case of equality is then obvious. $\square$

7 Spectra of flat 2-tori

In this section we recall the well-known formulas for the spectrum of the Laplacian and of the Dirac operator on flat 2-tori.

Because it is clear how the eigenvalues change under rescaling we will restrict to the case

$$T^2 = \frac{\mathbb{R}^2}{\Gamma_{xy}} \qquad \Gamma_{xy} = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\}, \quad y > 0$$

where T^2 carries the metric g_0 induced by the Euclidean metric of $\mathbb{R}^2$. The dual lattice $\Gamma_{xy}^* := H^1(T^2, \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(\Gamma_{xy}, \mathbb{Z})$ is generated by the vectors

$$\gamma_1 := \begin{pmatrix} 1 \\ -x/y \end{pmatrix} \quad \text{und} \quad \gamma_2 := \begin{pmatrix} 0 \\ 1/y \end{pmatrix}.$$

The function

$$f_\gamma : T^2 \to \mathbb{C} \quad f_\gamma(x) := \exp\left(2\pi i \langle \gamma, x \rangle\right) \quad \gamma \in \Gamma_{xy}^*$$

is an eigenfunction of the Laplace operator Δ on complex valued functions to the eigenvalue $4\pi^2|\gamma|^2$ where $|\,.\,|$ denotes the Euclidean norm on $\mathbb{R}^2$. Moreover, the family $(f_\gamma|\gamma \in \Gamma^*_{xy})$ is a complete system of eigenfunctions. Note that γ can also be viewed as a 1-form on T^2 and if $\|\,.\,\|_{L^2}$ is the L^2-norm defined in the previous section then

$$\|\gamma\|^2_{L^2} = |\gamma|^2 \text{area}.$$

Therefore we obtain

PROPOSITION 7.1. *The spectrum of the Laplacian on T^2 is given by the family*

$$\left\{ \frac{4\pi^2\,\|\gamma\|^2_{L^2}}{\text{area}} \,\Big|\, \gamma \in \Gamma^*_{xy} \right\}$$

where each eigenvalue appears with the correct multiplicity.

The first three eigenvalues can be easily expressed using the invariants of the previous section

$$\lambda_0 = 0 \qquad \lambda_1 = \lambda_2 = \frac{4\pi^2\,\sigma_1^2}{\text{area}}.$$

The eigenfunctions and eigenvalues of the square of the Dirac operator are very similar if the spin structure is trivial. Let ψ_1 and ψ_2 be parallel orthonormal spinors on T^2, then $(f_\gamma \psi_j | j = 1, 2; \gamma \in \Gamma^*_{xy})$ is a complete system of eigenfunctions to the eigenvalues $4\pi^2|\gamma|^2$. Therefore the eigenvalues $\mu_1 \le \mu_2 \le \mu_3 \ldots$ are the same as for the Laplace operator, but the multiplicities are doubled. In particular

$$\mu_1 = \mu_2 = 0 \qquad \mu_3 = \mu_4 = \mu_5 = \mu_6 = \frac{4\pi^2\,\sigma_1^2}{\text{area}}.$$

Now we assume that T^2 carries a non-trivial spin structure. After a rescaling of the metric and an orthonormal transformation of $\mathbb{R}^2$ we can assume that the spin structure is trivial on $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and non-trivial on $\begin{pmatrix} x \\ y \end{pmatrix}$ and that

$$0 \le x \le \frac{1}{2}, \qquad x^2 + \left(y - \frac{1}{2}\right)^2 \ge \frac{1}{4}, \qquad y > 0. \tag{6}$$

The set of all (x, y) satisfying (6) is called the *spin-conformal moduli space* $\mathcal{M}^{\text{spin}}$. The elements of $\mathcal{M}^{\text{spin}}$ correspond to equivalence classes of tori with non-trivial spin structures under the equivalence relation of conformal diffeomorphisms preserving the spin structure.

Let (ψ_1, ψ_2) be a basis of parallel sections of the spinor bundle on $\mathbb{R}^2$ and assume that they are pointwise orthogonal. Then

$$\Psi_{j,\gamma} := \exp\Big(2\pi i\,\langle \gamma, x\rangle\Big)\,\psi_j, \quad \gamma \in \Gamma^*_{xy} + \frac{\gamma_2}{2}$$

is a spinor that is invariant under the action of Γ_{xy}. Thus it defines an eigenspinor for $D^2 : \Sigma T^2 \to \Sigma T^2$ with eigenvalue $4\pi^2|\gamma|^2$ and the family $(\Psi_{j,\gamma}|j = 1, 2; \gamma \in \Gamma^*_{xy} + (\gamma_2/2))$ is a complete system of eigenspinors.

We obtain a similar proposition as above.

PROPOSITION 7.2 ([Fri84]). *Assume that T^2 carries a non-trivial spin structure as above. Then the spectrum of the square of the Dirac operator D^2 on T^2 is given by the family*

$$\left\{ \frac{4\pi^2 \, \|\gamma\|^2_{L^2}}{\text{area}} \,\Big|\, \gamma \in \Gamma^*_{xy} + \frac{\gamma_2}{2} \right\},$$

and the multiplicity of each eigenvalue in the spectrum of D^2 is twice the multiplicity in the family.

We want to prove that $\Gamma^*_{xy} + (\gamma_2/2)$ contains no vector that is shorter than $\gamma_2/2$. For this we need a lemma.

LEMMA 7.3. *If linearly independent vectors $v_1, v_2 \in \mathbb{R}^2$ satisfy*

$$0 \leq \langle v_1, v_2 \rangle \leq |v_1|^2 \leq |v_2|^2,$$

*then for any integers a, b with $a \neq 0$ **and** $b \neq 0$ the following inequality holds*

$$|av_1 + bv_2| \geq |v_2 - v_1|.$$

If $|av_1 + bv_2| = |v_2 - v_1|$, then $|a| = |b| = 1$.

Proof of lemma. Let $|av_1 + bv_2| \leq |v_2 - v_1|$. Without loss of generality we can assume that a and b are relatively prime. We obtain

$$a^2|v_1|^2 - 2\,|ab| \cdot \langle v_1, v_2 \rangle + b^2|v_2|^2 \leq |v_1|^2 - 2\,\langle v_1, v_2 \rangle + |v_2|^2$$

and therefore

$$\begin{aligned} (a^2 + b^2 - 2)\,|v_1|^2 &\leq (a^2 - 1)\,|v_1|^2 + (b^2 - 1)\,|v_2|^2 \\ &\leq 2(|ab| - 1)\,\langle v_1, v_2 \rangle \leq 2(|ab| - 1)\,|v_1|^2. \end{aligned}$$

Thus $(|a| - |b|)^2 \leq 0$ holds, i. e. $|a| = |b|$, and as we assumed that a and b are relatively prime we obtain $|a| = |b| = 1$. Because of $|v_1 + v_2| \geq |v_2 - v_1|$ the lemma holds. □

COROLLARY 7.4. *If $(x, y) \in \mathcal{M}^{\text{spin}}$, then:*

(a) There is no vector in $\Gamma^*_{xy} + (\gamma_2/2)$ that is shorter than $\gamma_2/2$.

(b) The shortest vectors in $\Gamma_{xy}^* - \{0\}$ *have length*

$$\min\left\{\frac{1}{y}, \frac{\sqrt{x^2+y^2}}{y}\right\}.$$

Proof.

(a) Because of relations (6) the vectors $v_1 := \gamma_1/2$ and $v_2 := (\gamma_1 + \gamma_2)/2$ satisfy the conditions of the lemma. Any element γ of $\Gamma_{xy}^* + (\gamma_2/2)$ can be written as $av_1 + bv_2$, $a, b \in \mathbb{Z} - \{0\}$. The lemma yields

$$|\gamma| \geq |v_2 - v_1| = \frac{|\gamma_2|}{2}.$$

(b) This time we set $v_1 = \gamma_1$ and $v_2 = \gamma_1 + \gamma_2$. As before $0 \leq \langle v_1, v_2\rangle \leq |v_1|^2 \leq |v_2|^2$. Any $\gamma \in \Gamma_{xy}^* - \{0\}$ is either a multiple of v_1 or v_2 (then $|\gamma|^2 \geq |v_1|^2 = |\gamma_1|^2 = 1 + (x^2/y^2)$) or

$$|\gamma| \geq |v_2 - v_1| = \frac{1}{y}.$$

□

Thus the smallest eigenvalue μ_1 of D^2 satisfies

$$\mu_1 = \pi^2|\gamma_2|^2 = \frac{\pi^2}{y^2}. \tag{7}$$

Using the notations of the previous section we see easily that the L^2-norm of the spin-structure χ satisfies

$$\|\chi\|_{L^2}^2 = \frac{1}{4y}.$$

With $\mathrm{area} = y$ we obtain

$$\mu_1\,\mathrm{area} = 4\pi^2\,\|\chi\|_{L^2}^2\,.$$

Analogously, we see for the cosystole that

$$\sigma_1^2 = \min\left\{\frac{1}{y}, \frac{x^2+y^2}{y}\right\}.$$

8 Regular bipartitions of 2-tori

Definition. A *regular bipartition* of T^2 is a pair (X_1, X_2) of disjoint open subsets $X_i \subset T^2$ such that $\partial X_1 = \partial X_2$ is a smooth 1-manifold, i. e. $\partial X_1 = \partial X_2$ is a disjoint union of finitely many smooth circles. In particular this implies $T^2 = X_1 \dot{\cup} X_2 \dot{\cup} \partial X_1$.

PROPOSITION 8.1. *Let (X_1, X_2) be a regular bipartition of T^2. Then exactly one of the following conditions is satisfied*

(i) The inclusion $X_1 \to T^2$ induces the trivial map $\pi_1(X_1) \to \pi_1(T^2)$.

(ii) The inclusion $X_2 \to T^2$ induces the trivial map $\pi_1(X_2) \to \pi_1(T^2)$.

(iii) The boundary ∂X_1 has at least two components that are non-contractible in T^2.

Proof. Assume a regular bipartition (X_1, X_2) satisfies (iii), then ∂X_1 contains a non-contractible loop. By a small perturbation we can achieve that this loop lies completely in X_1. Therefore $\pi_1(X_1) \to \pi_1(T^2)$ is not trivial. Hence (X_1, X_2) does not satisfy (i). Similarly we prove that it does not satisfy (ii).

Now assume that a regular bipartition (X_1, X_2) satisfies both (i) and (ii). Van-Kampen's theorem implies $\pi_1(T^2) = 0$. Therefore we have shown that at most one of the three conditions is satisfied.

It remains to show that at least one condition is satisfied. For this we assume that neither (i) nor (ii) is satisfied, i. e. there are continous paths $c_i : S^1 \to X_i$ that are non-contractible within T^2. Obviously ∂X_1 is homologous to zero. We will show that at least one component of ∂X_1 is non-homologous to zero. Then there has to be a second component that is non-homologous to zero, because $[\partial X_1] = 0$ is the sum of the homology classes of the components.

We argue by contradiction. Assume that each component of ∂X_1 is homologous to zero. Let $\pi : \mathbb{R}^2 \to T^2$ be the universal covering. Then $\pi^{-1}(\partial X_1)$ is diffeomorphic to a disjoint union of countably many S^1. We write

$$\pi^{-1}(\partial X_1) = \dot{\bigcup_{i \in \mathbb{N}}} Y_i$$

with $Y_i \cong S^1$. We choose lifts $\tilde{c}_i : \mathbb{R} \to \mathbb{R}^2$ of c_i, i. e. $\pi\left(\tilde{c}_i(t+z)\right) = c_i(t)$ for all $t \in [0,1]$, $z \in \mathbb{Z}$ and $i = 1, 2$. Then we take a path $\tilde{\gamma} : [0,1] \to \mathbb{R}^2$ joining $\tilde{c}_1(0)$ to $\tilde{c}_2(0)$. We can assume that $\tilde{\gamma}$ is transversal to any Y_i. We define I to be the set of all $i \in \mathbb{N}$ such that Y_i meets the trace of $\tilde{\gamma}$. The set I is finite. Using the Theorem of Jordan and Schoenfliess about simple closed curves in $\mathbb{R}^2$ we can inductively construct a compact set $K \subset \mathbb{R}^2$ with boundary $\bigcup_{i \in I} Y_i$. The number of intersections of $\tilde{\gamma}$ with $\bigcup_{i \in I} Y_i$ is odd.

Thus, either $\tilde{c}_1(0)$ or $\tilde{c}_2(0)$ is in the interior of K. But if $\tilde{c}_i(0)$ is in the interior of K, then the whole trace $\tilde{c}_i(\mathbb{R})$ is contained in K. Furthermore, $\tilde{c}_i(\mathbb{R}) = \pi^{-1}\left(c_i([0,1])\right)$ is closed and therefore compact. This implies that c_i is homologous to zero in contradiction to our assumption. □

9 Controling the conformal scaling function

Let T^2 carry an arbitrary metric g. According to the uniformization theorem we can write $g = e^{2u} g_0$ with a real function $u : T^2 \to \mathbb{R}$ and a flat metric g_0. The function u is unique up to adding a constant.

The aim of this section is to estimate the quantity $\operatorname{osc} u := \max u - \min u$. The estimate is similar to an estimate of the author in a previous publication [Amm00, Theorem 3.1]. The main difference is that the previous estimate needed the assumption

$$\|K_g\|_{L^p(T^2,g)} \Big(\operatorname{area}(T^2,g)\Big)^{1-(1/p)} < 4\pi$$

which is no longer needed in the estimate presented here.

THEOREM 9.1. *We assume*

$$\|K_g\|_{L^1(T^2,g)} < 4\pi.$$

Then for any $p \in \ ]1,\infty[$ we obtain a bound for the oscillation of u

(a) $\quad \operatorname{osc} u \le \mathcal{S}\Big(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)} \left(\operatorname{area}(T^2,g)\right)^{1-(1/p)}, p, \sigma_1(T^2,g)^{-2}\Big),$

(b) $\quad \operatorname{osc} u \le \mathcal{S}\Big(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)} \left(\operatorname{area}(T^2,g)\right)^{1-(1/p)}, p, \frac{\operatorname{area}(T^2,g)}{\operatorname{sys}_1(T^2,g)^2}\Big),$

where we use the definition

$$\mathcal{S}(\mathcal{K}_1,\mathcal{K}_p,p,\mathcal{V}) := \frac{p}{p-1}\left[\frac{\mathcal{K}_p}{4\pi} + \frac{1}{2}\left|\log\left(1-\frac{\mathcal{K}_1}{4\pi}\right)\right| + \frac{\mathcal{K}_1}{8\pi - 2\mathcal{K}_1}\log\left(\frac{2\mathcal{K}_p}{\mathcal{K}_1}\right)\right] + \frac{\mathcal{K}_1\mathcal{V}}{8}$$

for $\mathcal{K}_1 > 0$ and $\mathcal{S}(0,\mathcal{K}_p,p,\mathcal{V}) := 0$.

The function $\mathcal{S}$ is continuous in $\mathcal{K}_1 = 0$.

COROLLARY 9.2. *Let $\mathcal{F}$ be a family of Riemannian metrics conformal to the flat metric g_0. Assume that there are constants $\mathcal{K}_1 \in \ ]0,4\pi[$ and $\mathcal{K}_p \in \ ]0,\infty[$, $p \in \ ]1,\infty[$ with*

$$\|K_g\|_{L^1(T^2,g)} \le \mathcal{K}_1 \text{ and } \|K_g\|_{L^p(T^2,g)} \Big(\operatorname{area}(T^2,g)\Big)^{1-\frac{1}{p}} \le \mathcal{K}_p \text{ for any } g \in \mathcal{F}.$$

Then the oscillation $\operatorname{osc} u_g$ of the scaling function corresponding to g is uniformly bounded on $\mathcal{F}$ by

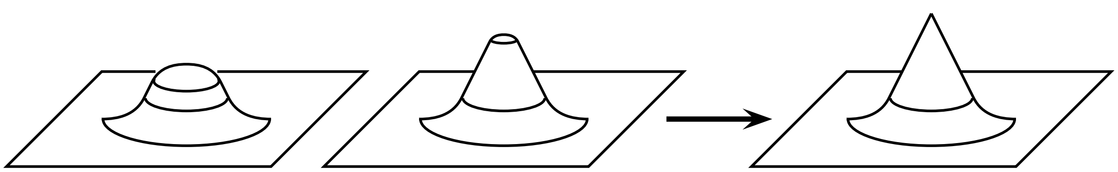
$$\operatorname{osc} u_g \le \mathcal{S}\Big(\mathcal{K}_1,\mathcal{K}_p,p,\mathcal{V}(T^2,g_0)\Big).$$

Before proving the theorem we will present some examples showing that the theorem and the corollary no longer hold if we drop one of the assumptions $\|K_g\|_{L^1(T^2,g)} \leq \mathcal{K}_1 < 4\pi$ or $\|K_g\|_{L^p(T^2,g)} (\mathrm{area}(T^2,g))^{1-\frac{1}{p}} \leq \mathcal{K}_p$.

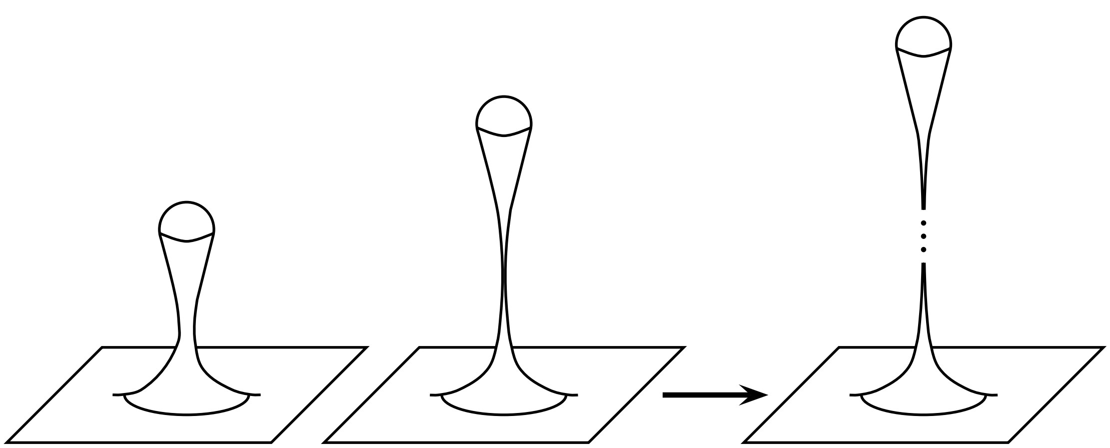
Example. For any $\mathcal{K}_1 > 0$ there is a sequence (g_i) of Riemannian metrics with fixed conformal type, bounded volume, constant systole, with

$$\|K_{g_i}\|_{L^1(T^2,g_i)} \leq \mathcal{K}_1 \text{ and } \operatorname{osc} u_{g_i} \to \infty.$$

In order to construct such a sequence we take a flat torus and replace a ball by a rotationally symmetric surface which approximates a cone for $i \to \infty$ (see [Amm00] for details).



Example. For any $\varepsilon > 0$ there is a sequence (g_i) of Riemannian metrics with fixed conformal type, bounded volume, constant systole, $-1 \leq K_{g_i} \leq 1$, $\|K_{g_i}\|_{L^1(T^2,g_i)} \leq 4\pi + \varepsilon$, $\|K_{g_i}\|_{L^p(T^2,g_i)} \leq$ *const* and $\operatorname{osc} u_{g_i} \to \infty$. In order to construct such a sequence we take a ball out of a flat torus and replace it by a hyperbolic part, a cone of small opening angle, and a cap as indicated in the following picture. While the injectivity radius of the hyperbolic part shrinks to zero, the oscillation of u tends to infinity.

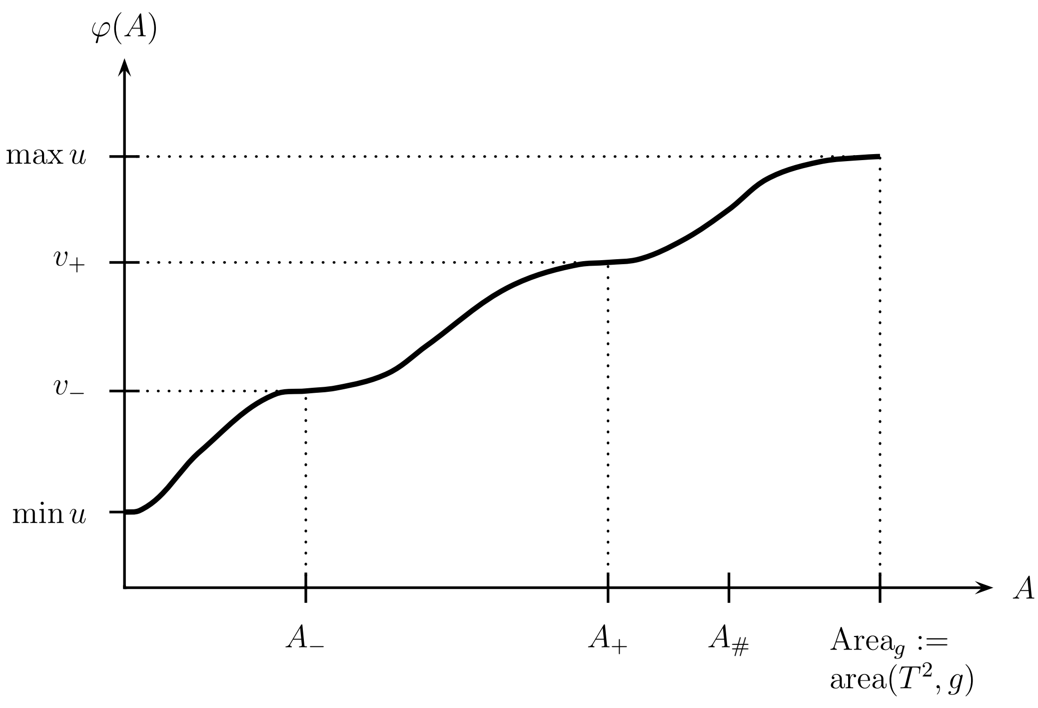


In the picture the dots in the “limit space” indicate the hyperbolic part with injectivity radius tending to 0 and diameter tending to ∞.

Proof of Theorem 9.1. As Morse functions form a dense subset of the space of C^∞-functions with respect to the C^∞-topology, we can assume without loss of generality that u is a Morse function. We set $\mathrm{Area}_g := \mathrm{area}(T^2, g)$ and $\mathrm{Area}_0 := \mathrm{area}(T^2, g_0)$. We define

$$G_<(v) := \left\{x \in T^2 \,|\, u(x) < v\right\} \qquad G_>(v) := \left\{x \in T^2 \,|\, u(x) > v\right\}$$

$$\begin{aligned} \varphi : [0, \mathrm{Area}_g] \;&\to\; \mathbb{R} \\ A \;&\mapsto\; \inf\left\{\sup_{x\in X} u(x) \,\Big|\, X \subset T^2 \text{ open},\ \mathrm{area}(X) \geq A\right\} && (8) \\ &= \;\sup\left\{\inf_{x\in X^c} u(x) \,\Big|\, X^c \subset T^2 \text{ open},\ \mathrm{area}(X^c) \geq \mathrm{Area}_g - A\right\} && (9) \end{aligned}$$



The infimum in (8) is actually a minimum and as u is a Morse function the only minimum is attained for $X = G_<(\varphi(A))$. Similarly the supremum in (9) is attained exactly in $X^c = G_>(\varphi(A))$. The function φ is strictly increasing and is continously differentiable. The inverse of φ is given by

$$\varphi^{-1}(v) = \mathrm{area}(G_<(v)).$$

The differential $\varphi'(A)$ is zero if and only if $\varphi(A)$ is a critical value of u.

Now let $v \in [\min u, \max u]$ be a regular value of u. We obtain

$$\left(\varphi^{-1}\right)'(v) \quad = \quad \int_{\partial G_<(v),g} \frac{1}{|du|_g} \geq \frac{\text{length}(\partial G_<(v),g)^2}{\int_{\partial G_<(v),g} |du|_g} \tag{10}$$

where length$(\partial G_<(v),g)$ is the length of the boundary of $\partial G_<(v)$ with respect to g. This inequality will yield an upper bound for φ' which will provide in turn an upper bound for $\operatorname{osc} u = \varphi(\text{Area}_g) - \varphi(0) = \int_0^{\text{Area}_g} \varphi'$. We transform

$$\int_{\partial G_<(v),g} |du|_g = \int_{\partial G_<(v)} * \, du = -\int_{G_<(v),g} \Delta_g u = -\int_{G_<(v),g} K_g. \tag{11}$$

The last equation follows from the Kazdan-Warner-equation $\Delta_g u = K_g$ [KW74]. We define κ using the Gaussian curvature function $K_g : T^2 \to \mathbb{R}$

$$\kappa : [0, \text{Area}_g] \to \mathbb{R}, \quad \kappa(A) := \inf \left\{ \sup_{x \in X} K_g(x) \,\Big|\, X \subset T^2 \text{ open}, \ \text{area}(X) \geq A \right\}.$$

Any open subset $X \subset T^2$ satisfies

$$\int_0^{\text{area}(X,g)} \kappa \leq \int_{X,g} K_g \leq \int_{\text{Area}_g - \text{area}(X,g)}^{\text{Area}_g} \kappa$$

and for $X = T^2$ we have equality. Using Gauss-Bonnet theorem we see that

$$\int_0^{\text{Area}_g} \kappa = 0.$$

The right hand side of equation (11) now can be estimated as follows.

$$-\int_{G_<(\varphi(A)),g} K_g \leq -\int_0^A \kappa = \int_A^{\text{Area}_g} \kappa \tag{12}$$

Putting (10), (11) and (12) together, we obtain

$$\varphi'(A) \leq \frac{\int_A^{\text{Area}_g} \kappa}{\text{length}(\partial G_<(\varphi(A)),g)^2}.$$

Our next goal is to find suitable lower bounds for length$(\partial G_<(\varphi(A))$.

Note that for any regular value v of u, $(G_<(v), G_>(v))$ is a regular bipartition of T^2. According to Proposition 8.1 exactly one of the following conditions is satisfied

(i) The inclusion $G_<(v) \to T^2$ induces the trivial map $\pi_1(G_<(v)) \to \pi_1(T^2)$.

(ii) The inclusion $G_>(v) \to T^2$ induces the trivial map $\pi_1(G_>(v)) \to \pi_1(T^2)$.

(iii) The boundary $\partial G_<(v)$ has at least two components that are non-contractible in T^2.

If condition (i) is satisfied by v, it is obvious that it is also satisfied by $v' \in [0, v]$. Similarly, if condition (ii) is satisfied by v, then it is also satisfied by $v' \in [v, \mathrm{Area}_g]$.

$$\begin{aligned} v_- &:= \sup\{v \in [0, \mathrm{Area}_g] \,|\, \text{(i) is satisfied for } v\} \\ v_+ &:= \inf\{v \in [0, \mathrm{Area}_g] \,|\, \text{(ii) is satisfied for } v\} \\ A_\pm &:= \varphi^{-1}(v_\pm). \end{aligned}$$

In each of the three cases we derive a different estimate for length$(\partial G_<(v), g)$ and therefore we obtain a different bound for φ'.

(i) In this case $G_<(v)$ can be lifted to the universal covering $\mathbb{R}^2$ of T^2. We will also write g and g_0 for the pullbacks of g and g_0 to $\mathbb{R}^2$. The isoperimetric inequality of the flat space $(\mathbb{R}^2, g_0)$ yields

$$\text{length}(\partial G_<(v), g_0)^2 \geq 4\pi \,\mathrm{area}(G_<(v), g_0).$$

Using the relations

$$\begin{aligned} \text{length}(\partial G_<(v), g) &= e^v \,\text{length}(\partial G_<(v), g_0) && (13) \\ \mathrm{area}(G_<(v), g) &\leq e^{2v} \,\mathrm{area}(G_<(v), g_0) && (14) \end{aligned}$$

we obtain

$$\text{length}(\partial G_<(v), g)^2 \geq 4\pi \,\mathrm{area}(G_<(v), g). \tag{15}$$

Together with the Hölder inequality

$$-\int_0^A \kappa \leq \|K_g\|_{L^p(T^2,g)} A^{1-(1/p)}$$

we get

$$\begin{aligned} \varphi'(A) = \frac{1}{(\varphi^{-1})'(\varphi(A))} &\leq \frac{-\int_0^A \kappa}{\text{length}(\partial G_<(\varphi(A)), g)^2} \\ &\leq \frac{1}{4\pi} \|K_g\|_{L^p(T^2,g)} A^{-\frac{1}{p}} \end{aligned}$$

Integration yields

$$\begin{aligned} v_- - \min u &= \varphi(\varphi^{-1}(v_-)) - \varphi(0) \\ &\leq \frac{p}{p-1} \frac{1}{4\pi} \|K_g\|_{L^p(T^2,g)} (\varphi^{-1}(v_-))^{1-(1/p)} \\ &\leq \frac{p}{p-1} \frac{1}{4\pi} \|K_g\|_{L^p(T^2,g)} (\mathrm{Area}_g)^{1-(1/p)} && (16) \end{aligned}$$

(ii) This case is similar to the previous one, but unfortunately because of opposite signs some estimates do not work as before. For example (14) and (15) are no longer true for $G_<(v)$ replaced by $G_>(v)$. Instead we use Topping's inequality [Top98a, Top99].

$$(\text{length}(\partial G_>(v), g))^2 \geq 4\pi\, \hat{A} - 2\int_0^{\hat{A}} (\hat{A} - a)\, \kappa(\text{Area}_g - a)\, da \tag{17}$$

with $\hat{A} = \text{area}(G_>(v), g)$. Using the estimate

$$\begin{aligned} \int_0^{\hat{A}} (\hat{A} - a)\, \kappa(\text{Area}_g - a)\, da &\leq \hat{A} \int_0^{\hat{A}} \max\{0, \kappa(\text{Area}_g - a)\}\, da \\ &\leq \frac{\hat{A}}{2}\, \|K_g\|_{L^1(T^2,g)} \end{aligned}$$

we obtain

$$(\text{length}(\partial G_>(v), g))^2 \geq \left(4\pi - \|K_g\|_{L^1(T^2,g)}\right)\, \hat{A}. \tag{18}$$

The obvious inequality

$$\int_{\text{Area}_g - \hat{A}}^{\text{Area}_g} \kappa \leq \|\max\{0, K_g\}\|_{L^1(T^2,g)} \leq (1/2)\, \|K_g\|_{L^1(T^2,g)}$$

yields

$$\varphi'(\text{Area}_g - \hat{A}) \leq \frac{1}{\hat{A}}\, \frac{\|K_g\|_{L^1(T^2,g)}}{8\pi - 2\,\|K_g\|_{L^1(T^2,g)}}.$$

Integration yields

$$\varphi(\text{Area}_g - \hat{A}) - \varphi(A_+) \leq \log\left(\frac{\text{Area}_g - A_+}{\hat{A}}\right)\, \frac{\|K_g\|_{L^1(T^2,g)}}{8\pi - 2\,\|K_g\|_{L^1(T^2,g)}}.$$

The right hand side converges to ∞ for $\hat{A} \to 0$. Thus we have to improve our estimates for small $\hat{A}$. The integral in (17) also has the following bound.

$$\begin{aligned} \int_0^{\hat{A}} (\hat{A} - a)\, \kappa(\text{Area}_g - a)\, da &\leq \left(\int_0^{\hat{A}} (\hat{A} - a)^q\, da\right)^{1/q} \cdot \left(\int_0^{\hat{A}} \left|\kappa(\text{Area}_g - a)\right|^p da\right)^{1/p} \\ &= \left(\frac{\hat{A}^{q+1}}{q+1}\right)^{1/q} \cdot \|K_g\|_{L^p(T^2,g)} \end{aligned} \tag{19}$$

where we wrote $q := p/(p-1)$ in order to simplify the notation.

We obtain a second lower bound on the length

$$(\text{length}(\partial G_>(v), g))^2 \geq 4\pi\, \hat{A} - c\hat{A}^{1+\frac{1}{q}}\, \|K_g\|_{L^p(T^2,g)} \tag{20}$$

for any $c \geq 2/\sqrt[q]{q+1}$, e. g. $c = 2$. Note that our assumption $\|K_g\|_{L^1(T^2,g)} < 4\pi$ does not imply that the right hand side of the above inequality is always positive. Although (20) is better for small $\hat{A}$, it is not strong enough to control the length for larger $\hat{A}$. However, for

$$\hat{A} < \left(\frac{4\pi}{c \cdot \|K_g\|_{L^p(T^2,g)}} \right)^q$$

we use (20) and

$$\int_{\mathrm{Area}_g - \hat{A}}^{\mathrm{Area}_g} \kappa \leq \hat{A}^{1/q} \, \|K_g\|_{L^p(T^2,g)}$$

to obtain the estimate

$$\varphi'(\mathrm{Area}_g - \hat{A}) \quad \leq \quad \frac{\hat{A}^{-1/p} \, \|K_g\|_{L^p(T^2,g)}}{4\pi - c\hat{A}^{1/q} \|K_g\|_{L^p(T^2,g)}}.$$

With the substitution

$$w = w(A) = 4\pi - c(\mathrm{Area}_g - A)^{1/q} \|K_g\|_{L^p(T^2,g)}$$

integration yields

$$\begin{aligned} \varphi(\mathrm{Area}_g) - \varphi(A_\#) &= \int_{A_\#}^{\mathrm{Area}_g} \varphi'(A) \, dA \\ &\leq \int_{w(A_\#)}^{w(\mathrm{Area}_g)} \frac{q}{c} \frac{1}{w} \, dw = \frac{q}{c} \log \frac{w(\mathrm{Area}_g)}{w(A_\#)} \\ &= \frac{q}{c} \log \frac{4\pi}{4\pi - c(\mathrm{Area}_g - A_\#)^{1/q} \|K_g\|_{L^p(T^2,g)}} \end{aligned}$$

for any $A_\#$ between $\mathrm{Area}_g - \left(4\pi/(c \cdot \|K_g\|_{L^p(T^2,g)})\right)^q$ and Area_g. We choose

$$A_\# := \max \left\{ \mathrm{Area}_g - \left(\frac{\|K_g\|_{L^1(T^2,g)}}{2 \, \|K_g\|_{L^p(T^2,g)}} \right)^q, A_+ \right\}.$$

Finally we obtain the estimates

$$\max u - \varphi(A_\#) \quad \leq \quad \frac{q}{c} \log \frac{8\pi}{8\pi - c \|K_g\|_{L^1(T^2,g)}} \tag{21}$$

$$\varphi(A_\#) - v_+ \quad \leq \quad q \, \frac{\|K_g\|_{L^1(T^2,g)}}{8\pi - 2 \|K_g\|_{L^1(T^2,g)}} \, \log \left(\frac{2 \, {\mathrm{Area}_g}^{1/q} \, \|K_g\|_{L^p(T^2,g)}}{\|K_g\|_{L^1(T^2,g)}} \right) \tag{22}$$

For $c = 2$ the right hand sides of these inequalities contribute two summands to the formula for $\mathcal{S}$.

(iii) If $v = \varphi(A)$ is a regular value of u between v_- and v_+, then $\partial G_<(v)$ contains at least two components that are non-contractible in T^2. Hence, for any metric $\tilde{g}$ on T^2 we get

$$\text{length}(\partial G_<(v), \tilde{g}) \geq 2\,\text{sys}_1(T^2, \tilde{g}).$$

In order to prove (a) of Theorem 9.1 we apply this equation to $\tilde{g} := g_0$. Using $\int_A^{\text{Area}_g} \kappa \leq (1/2)\,\|K_g\|_{L^1(T^2,g)}$ and $\text{length}(\partial G_<(v), g) = e^v\,\text{length}(\partial G_<(v), g_0)$ we obtain

$$\begin{aligned} \varphi'(A) &\leq e^{-2\varphi(A)} \frac{\int_A^{\text{Area}_g} \kappa}{4\,\text{sys}_1(T^2, g_0)^2} \\ &\leq \frac{1}{8}\, e^{-2\varphi(A)}\, \frac{\|K_g\|_{L^1(T^2,g)}}{\text{sys}_1(T^2, g_0)^2}. \end{aligned} \tag{23}$$

Integration yields

$$\begin{aligned} v_+ - v_- &= \int_{A_-}^{A_+} \varphi'(A)\, dA \\ &\leq \frac{1}{8}\, \frac{\|K_g\|_{L^1(T^2,g)}}{\text{sys}_1(T^2, g_0)^2} \int_{A_-}^{A_+} e^{-2\varphi(A)}\, dA \\ &\leq \frac{1}{8}\, \frac{\|K_g\|_{L^1(T^2,g)}}{\text{sys}_1(T^2, g_0)^2}\, \text{Area}_0 \end{aligned} \tag{24}$$

where we used $\text{Area}_0 = \text{area}(T^2, g_0) = \int_0^{\text{Area}_g} e^{-2\varphi(A)}\, dA$.

Together with inequalities (16), (21) and (22) we obtain statement (a) of the theorem.

Similarly, setting $\tilde{g} := g$ we get statement (b).

□

10 Some “inverse” inequalities

In Proposition 6.1 and Lemma 6.2 we proved some inequalities relating the metric g to g_0. It is easy to prove that they also hold in the other direction if we add a factor like $e^{2\,\text{osc}\,u}$.

Explicitely we obtain:

(a) $$\frac{\text{sys}_1(T^2, g)^2}{\text{area}(T^2, g)} \geq e^{-2\,\text{osc}\,u} \frac{\text{sys}_1(T^2, g_0)^2}{\text{area}(T^2, g_0)}$$

(b) $$\frac{\text{nonspin-sys}_1(T^2, g, \chi)^2}{\text{area}(T^2, g)} \geq e^{-2\,\text{osc}\,u} \frac{\text{nonspin-sys}_1(T^2, g_0, \chi)^2}{\text{area}(T^2, g_0)}$$

(c) $$\frac{\text{spin-sys}_1(T^2,g,\chi)^2}{\text{area}(T^2,g)} \geq e^{-2\operatorname{osc} u}\frac{\text{spin-sys}_1(T^2,g_0,\chi)^2}{\text{area}(T^2,g_0)}$$

(d) For any $\eta \in H^1(T^2,\mathbb{Z}_2)$ and $1 \leq p \leq 2 \leq q \leq \infty$

$$\begin{aligned}
\|\eta\|_{L^p(T^2,g)}\,\text{area}(T^2,g)^{\left(\frac{1}{2}-\frac{1}{p}\right)} &\geq e^{\left(1-\frac{2}{p}\right)\operatorname{osc} u}\,\|\eta\|_{L^p(T^2,g_0)}\,\text{area}(T^2,g_0)^{\left(\frac{1}{2}-\frac{1}{p}\right)}\\
\|\eta\|_{L^q(T^2,g)}\,\text{area}(T^2,g)^{\left(\frac{1}{2}-\frac{1}{q}\right)} &\leq e^{\left(1-\frac{2}{q}\right)\operatorname{osc} u}\,\|\eta\|_{L^q(T^2,g_0)}\,\text{area}(T^2,g_0)^{\left(\frac{1}{2}-\frac{1}{q}\right)}
\end{aligned}$$

A combination of these inequalities together with our upper bound for $\operatorname{osc} u$ in the previous section enables us to compare the quantities under consideration for a flat and an arbitrary metric in the same (spin-)conformal class.

11 Proof of the main results

Combining the inequalities derived in the previous sections, we are now able to derive our main results.

Theorem 2.1 is a consequence of Proposition 7.2 together with Proposition 5.2 and Theorem 9.1. Theorem 2.2 then follows from the calculation of the first eigenvalue of D^2 on flat tori at the end of section 7. Using the inequalities in Proposition 6.1 and section 10 we can derive Corollaries 2.3 and 2.4.

Similarly, Theorem 2.5 is a consequence of Proposition 7.1 together with Proposition 5.1 and Theorem 9.1. Theorem 2.6 then follows from the calculation of the first positive eigenvalue of Δ on flat tori at the end of section 7. Using the inequalities in Proposition 6.1 we obtain Corollary 2.7.

12 An application to the Willmore functional

In this section S^3 always carries the metric g_{S^3} of constant sectional curvature 1. For any immersion $F: T^2 \to S^3$ we define the Willmore functional

$$\mathcal{W}(F) := \int_{(T^2,F^*g_{S^3})} |H_{T^2\to S^3}|^2 + 1$$

where H is the relative mean curvature of $F(T^2)$ in S^3 and integration is the usual integration of functions $T^2 \to \mathbb{R}$ over the Riemannian manifold $(T^2, F^*g_{S^3})$. Note that the mean curvature H of $F(T^2)$ in $\mathbb{R}^4$ satisfies

$$|H|^2 = |H_{T^2\to S^3}|^2 + 1.$$

The Willmore conjecture states that $\mathcal{W}(F) \geq 2\pi^2$. Li and Yau [LY82, Fact 3] proved that the conjecture holds if F is not an embedding.

Any immersion $F : T^2 \to S^3$ induces a spin structure φ_F on T^2. The spin structure φ_F is non-trivial if and only if F is regularly homotopic to an embedding. Thus for any immersion F which is regularly homotopic to an embedding, the pair $(F^* g_{S^3}, \varphi_F)$ defines an element (x, y) in the spin-conformal moduli space $\mathcal{M}^{\text{spin}}$ (defined in section 7). In order to shorten our notation we write $[F] := (x, y) \in \mathcal{M}^{\text{spin}}$. If T^2 already carries a spin structure, we say that F is *spin* iff $\varphi_F = \varphi$.

Li and Yau proved:

THEOREM 12.1 ([LY82, Theorem 1]). *Let $F : (T^2, g) \to (S^3, g_{S^3})$ be a conformal embedding, let Area_g be the area of (T^2, g) and let λ_1 be the first positive eigenvalue of the Laplacian Δ on (T^2, g) then*

$$\mathcal{W}(F) \geq \frac{1}{2}\,\lambda_1 \mathrm{Area}_g.$$

From this theorem the conjectured inequality $\mathcal{W}(F) \geq 2\pi^2$ follows, if $[F]$ lies in a compact subset of $\mathcal{M}^{\text{spin}}$ with positive measure (see Figure 1).

A similar lower bound for $\mathcal{W}(F)$ in terms of Dirac eigenvalues has been given by Bär.

THEOREM 12.2 ([Bär98b]). *Let $F : (T^2, g, \varphi) \to (S^3, g_{S^3})$ be an **isometric** spin immersion. Then for the first eigenvalue μ_1 of the square of the Dirac operator the inequlity*

$$\mathcal{W}(F) \geq \mu_1 \mathrm{Area}_g$$

holds.

Note that this estimate is only non-trivial if F is regularly homotopic to an embedding.

Remark. At the end of this section we will show by example that in general "isometric" can not be replaced be "conformal" in this theorem.

Our goal now is to apply our previous estimates and derive lower bounds for $\mathcal{W}(F)$. One way to deduce such bounds is to combine the theorem with our lower estimates for the first eigenvalue of the square of the Dirac operator. These lower etimates for the Willmore functional are weaker than the ones derived by the author in [Amm00], therefore we skip this approach.

In this article, our approach is to modify the techniques of Theorem 12.2. This yields together with Theorem 9.1 new results about the Willmore functional.

As in the previous sections we define

$$\mathcal{S}(\mathcal{K}_1, \mathcal{K}_p, p, \mathcal{V}) := \frac{p}{p-1}\,\left[\frac{\mathcal{K}_p}{4\pi} + \frac{1}{2}\left|\log\left(1 - \frac{\mathcal{K}_1}{4\pi}\right)\right| + \frac{\mathcal{K}_1}{8\pi - 2\mathcal{K}_1}\,\log\left(\frac{2\mathcal{K}_p}{\mathcal{K}_1}\right)\right] + \frac{\mathcal{K}_1 \mathcal{V}}{8}$$

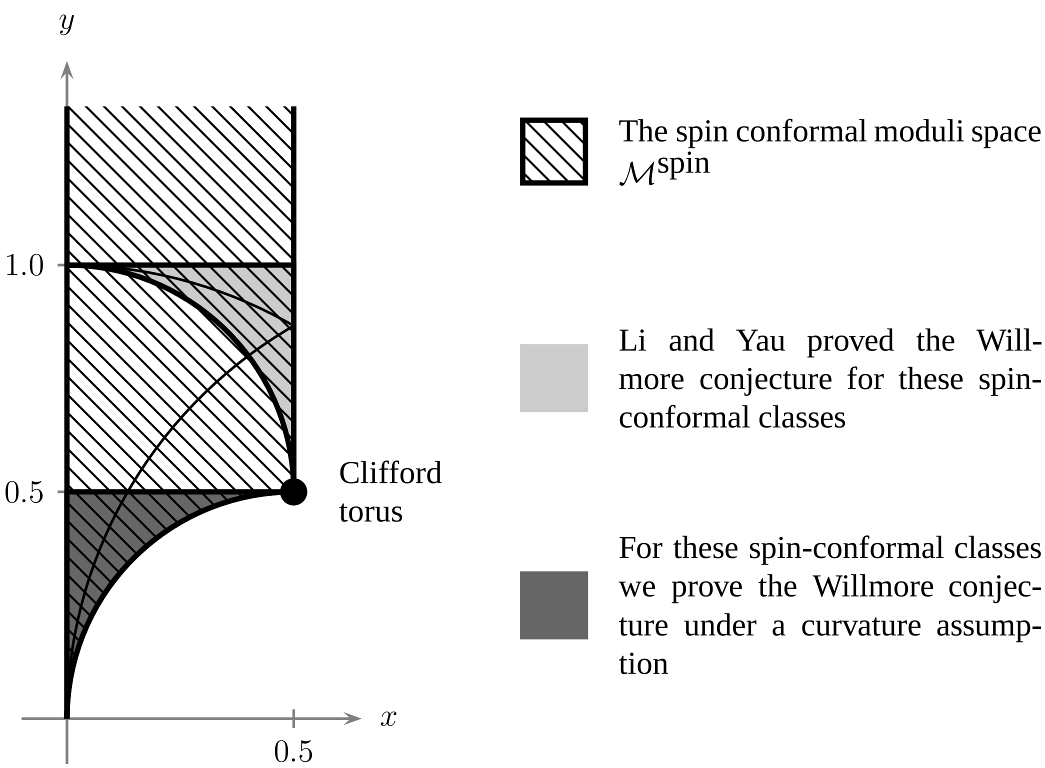


Figure 1: The spin conformal moduli space

for $\mathcal{K}_1 > 0$ and $\mathcal{S}(0, \mathcal{K}_p, p, \mathcal{V}) := 0$.

THEOREM 12.3. *Let $F : T^2 \to S^3$ be an immersion of the 2-dimensional torus in S^3 carrying the standard metric g_{S^3}. Let F be regularly homotopic to an embedding. We set $g := F^* g_{S^3}$. Let $(x, y) = [F] \in \mathcal{M}^{\text{spin}}$. Then*

$$\mathcal{W}(F) \geq \frac{\pi^2}{y} - \frac{1}{8} (\operatorname{osc} u) \, \|K_g\|_{L^1(T^2,g)} .$$

In particluar if $\|K_g\|_{L^1(T^2,g)} < 4\pi$ and any $p > 1$

$$\mathcal{W}(F) \geq \frac{\pi^2}{y} - \frac{1}{8} \mathcal{S} \, \|K_g\|_{L^1(T^2,g)}$$

with $\mathcal{S} := \mathcal{S}\left(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)} \operatorname{Area}_g{}^{1-(1/p)}, p, \frac{\operatorname{Area}_g}{\operatorname{sys}_1(T^2,g)^2} \right)$ or
$\mathcal{S} := \mathcal{S}\left(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)} \operatorname{Area}_g{}^{1-(1/p)}, p, \sigma_1(T^2, g)^{-2} \right)$.

Proof. We write the induced metric g on T^2 in the form $g = e^{2u} g_0$ with g_0 flat. Any Killing spinor on S^3 with the Killing constant $\alpha = (1/2)$ induces a spinor ψ on (T^2, g)

satisfying

$$D_g\psi = H\psi + \nu\psi,$$

where

$$\nu = \gamma(e_1)\gamma(e_2) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \in \mathrm{End}\left(\Sigma^+T^2 \oplus \Sigma^-T^2\right)$$

(see e. g. [Bär98b]). There is an isomorphism of vector bundles [Hit74],[Hij86, 4.3.1]

$$\begin{array}{ccc} \Sigma T^2 & \to & \widetilde{\Sigma} T^2 \\ \Psi & \mapsto & \widehat{\Psi} \end{array}$$

with

$$e^u\,\widehat{D_g\Psi} = D_{g_0}\widehat{\Psi} + \frac{1}{2}\gamma_{g_0}(\mathrm{grad}_{g_0}u)\widehat{\Psi}$$

and

$$|\widehat{\Psi}| = |\Psi|.$$

Here γ_{g_0} means Clifford multiplication corresponding to the metric g_0. Note that $\widetilde{\Psi}$ from section 5 satisfies $\widetilde{\Psi} = e^{(u/2)}\widehat{\Psi}$.

We apply this transformation for $\Psi = \psi$ and we obtain

$$D_{g_0}\widehat{\psi} = -\frac{1}{2}\gamma_{g_0}(\mathrm{grad}_{g_0}u)\widehat{\psi} + e^u H\widehat{\psi} + e^u\nu\widehat{\psi}.$$

As ν, $\gamma(V)$ and $\nu\gamma(V)$ are skew-hermitian for any vector V, this yields

$$\begin{aligned} \left|D_{g_0}\widehat{\psi}\right|^2 &= \frac{1}{4}\left|\gamma_{g_0}(\mathrm{grad}_{g_0}u)\widehat{\psi}\right|^2 + e^{2u}H^2\left|\widehat{\psi}\right|^2 + e^{2u}\left|\nu\widehat{\psi}\right|^2 \\ &= \frac{1}{4}|du|_{g_0}^2 + e^{2u}H^2 + e^{2u}. \end{aligned}$$

Integration over (T^2, g_0) provides

$$\tilde{\lambda}_1 \mathrm{area}(T^2, g_0) \le \frac{1}{4}\int_{T^2} |du|_{g_0}^2 \,\mathrm{dvol}_{g_0} + \mathcal{W}(F),$$

where $\tilde{\lambda}_1$ denotes the smallest eigenvalue of the square of the Dirac operator on (T^2, g_0).

On the other hand

$$\begin{aligned} \int_{T^2} |du|_{g_0}^2 \,\mathrm{dvol}_{g_0} &= \int_{T^2} u\Delta_{g_0}u \,\mathrm{dvol}_{g_0} \\ &= \int_{T^2} e^{2u}uK_g \,\mathrm{dvol}_{g_0} \\ &= \int_{T^2} uK_g \,\mathrm{dvol}_g \\ &\le \frac{1}{2}(\mathrm{osc}\, u)\, \|K_g\|_{L^1(T^2,g)}. \end{aligned}$$

Together with Theorem 9.1 and the results of section 7 we get the statement. □

COROLLARY 12.4. *For any $\kappa_1 \in\ ]0, 4\pi[$, any $p > 1$ and any $\kappa_p > 0$ there is a neighborhood U of the $(y \to 0)$-end of $\mathcal{M}^{\text{spin}}$ with the following property:*
If $F : T^2 \to S^3$ is an immersion such that the induced metric $g := F^* g_{S^3}$ and the induced spin structure φ_F represent a spin-conformal class in U and if the curvature conditions

$$\|K_g\|_{L^1(T^2,g)} < \kappa_1 \quad \textit{and} \quad \|K_g\|_{L^p(T^2,g)} \operatorname{Area}_g{}^{1-(1/p)} < \kappa_p$$

are satisfied, then the Willmore conjecture

$$W(F) \geq 2\pi^2$$

holds.

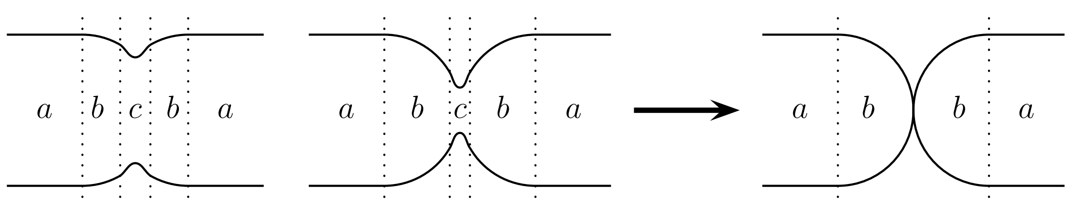
COROLLARY 12.5. *Let $F_i : T^2 \to S^3$ be a sequence of immersions. The induced metrics $g_i := F_i^* g_{S^3}$ together with the induced spin structures define a sequence (x_i, y_i) in the spin-moduli space $\mathcal{M}^{\text{spin}}$. Assume that $y_i \to 0$ and that the curvature conditions*

$$\|K_{g_i}\|_{L^1(T^2,g_i)} < \kappa_1 < 4\pi \quad \textit{and} \quad \|K_g\|_{L^p(T^2,g)} \operatorname{Area}_g{}^{1-(1/p)} < \kappa_p$$

are satisfied for some $p > 1$ and $\kappa_p < \infty$. Then

$$\mathcal{W}(F_i) \to \infty.$$

The conclusion of the second corollary is false if we drop the curvature conditions. To see this we construct a sequence of immersions with $y_i \to 0$ and $\mathcal{W}(F_i) < const$. We start with an embedding $F : T^2 \to S^3$ which looks in a neighborhood of some point like a cylinder. Now we "strangle" the torus as in the picture below:



We get a sequence $F_i : T^2 \to S^3$ of C^1-embeddings with the following properties:

(i) $F_i(T^2)$ coincides with $F(T^2)$ in region a

(ii) $F_i(T^2)$ coincides with a part of a half-sphere in region b,

(iii) $F_i(T^2)$ coincides with a minimal surface in region c.

Note that the regions a, b and c depend on i. In the limit $i \to \infty$, region c disappears. After smoothing we get a family of smooth embeddings satisfying both $y_i \to 0$ and $\mathcal{W}(F_i) <$ *const* and $\mathrm{area}(T^2, F_i^* g_{S^3}) \to$ *const*.

Hence, the first eigenvalue of D^2 is bounded from above. But the first eigenvalue of the spin-conformally equivalent flat torus with unit volume converges to ∞. This implies that there are spin-conformal classes in which the optimal constants in Lott's inequality (2) are not attained by flat metrics.

From this example we can also conclude that Theorem 12.2 does no longer hold, if we replace the condition "isometric spin immersion" by "conformal spin immersion".

13 Comparing spectra for different spin structures

In this section we remove the assumption $\dim M = 2$ and assume that a compact Riemannian spin manifold (M, g) of arbitrary dimension carries at least two different spin structures ϑ and ϑ'. The space of spin structures on M is an affine space associated to the vector space $H^1(M, \mathbb{Z}_2)$ which will be identified with $\mathrm{Hom}_{\mathbb{Z}}(H_1(M, \mathbb{Z}), \mathbb{Z}_2)$ and $\mathrm{Hom}(\pi_1(M), \mathbb{Z}_2)$.

For $r \in \mathbb{R}$, let $H^1_{\mathbb{R}}(M, r\mathbb{Z})$ be the set of all $[\omega] \in H^1_{\text{deRham}}(M, \mathbb{R})$ satisfying

$$\int_X \omega \in r\mathbb{Z} \quad \text{for any closed 1-chain } X.$$

Generalizing our definition in section 6 we define

$$\begin{aligned} P : H^1_{\mathbb{R}}\left(M, \frac{1}{2}\mathbb{Z}\right) &\to \mathrm{Hom}_{\mathbb{Z}}(H_1(M, \mathbb{Z}), \mathbb{Z}_2) = H^1(M, \mathbb{Z}_2) \\ [\omega] &\mapsto \left([X] \mapsto \exp(2\pi i \int_X \omega)\right). \end{aligned}$$

The kernel of P is $H^1_{\mathbb{R}}(M, \mathbb{Z})$. We now define the *stable norm* for elements of χ of $H^1(M, \mathbb{Z}_2)$

$$\|\chi\|_{L^\infty} := \inf \{\|\omega\|_{L^\infty} \mid P([\omega]) = \chi\}.$$

In general P is not surjective, hence this norm takes values in $[0, \infty]$. The elements in the image of P are called *realizable by a differentiable form*. A homomorphism $\chi \in \mathrm{Hom}_{\mathbb{Z}}(H_1(M, \mathbb{Z}), \mathbb{Z}_2)$ is realizable by a differentiable form if and only if χ vanishes on the torsion subgroup of $H_1(M, \mathbb{Z})$.

Definition. Two families $(\lambda_i | i \in \mathbb{Z})$ and $(\lambda'_i | i \in \mathbb{Z})$ of real numbers are said to be δ-*close* if there is a bijective map $h : \mathbb{Z} \to \mathbb{Z}$ with the property

$$|\lambda_{h(i)} - \lambda'_i| \leq \delta.$$

PROPOSITION 13.1. *Assume that (M,g) carries two spin structures whose difference χ is realizable as a differential form. Then the spectra of D for the two spin structures are $2\pi \|\chi\|_{L^\infty}$-close.*

Proof. We modify a technique used by Friedrich [Fri84] for calculating the spectrum of the Dirac operator on a flat torus.

Let us assume that the difference χ of the spin structures is realizable as a differentiable form. We take $\omega \in H^1_{\mathbb{R}}(M,(1/2)\mathbb{Z})$ with $P([\omega]) = \chi$ and $\|\omega\|_{L^\infty} \le \|\chi\|_{L^\infty} + \varepsilon$ for a small number $\varepsilon > 0$. Then there is complex line bundle L_ω on M which is trivialized by a section τ and a connection ∇ on L_ω such that

$$\nabla_X \tau = 2\pi i \omega(X)\tau.$$

The holonomy of the bundle (L_ω, ∇) is χ. Therefore the square of (L_ω, ∇) admits a parallel trivialization. Let L_ω carry the hermitian metric characterized by $|\tau| \equiv 1$.

Denote by ΣM and $\Sigma' M$ the spinor bundles to the two spin structures. Then

$$\Sigma' M \cong \Sigma M \otimes L_\omega$$

where the isomorphism preserves the connection, the hermitian metric and the Clifford multiplication. Now we define

$$\begin{array}{rcl} H : \Gamma(\Sigma M) & \to & \Gamma(\Sigma' M) \\ \Psi & \mapsto & \Psi \otimes \tau \end{array}$$

The Dirac operators D and D' for the two spin structures then satisfy

$$D'\Psi = H \circ D \circ H^{-1}\Psi + 2\pi i \omega \cdot \Psi$$

where $\cdot$ denotes the Clifford multiplication. Multiplication by $2\pi i\omega$ is a bounded operator on the space of L^2-sections of $\Sigma' M$. Its operator norm is $2\pi \|\omega\|_{L^\infty}$. The following well-known lemma completes the proof. □

LEMMA 13.2. *Let D and D' be two self-adjoint densely defined endomorphisms of a complex separable Hilbert space. We assume that the spectra of D and D' are discrete with finite multiplicities. Suppose that $D - D'$ is a bounded operator of operator norm K. Then the spectra of D and D' (with multiplicities) are K-close.*

The lemma is well-known in perturbation theory. For example it can be deduced from considerations in [Kat66]. The eigenvalues $(\lambda_i(t) \mid i \in \mathbb{Z})$ of

$$A_t := (1-t)D + tD', \quad t \in [0,1]$$

can be numbered such that $\lambda_i(t)$ is a Lipschitz function in t with Lipschitz constant K. From this observation the lemma is evident.

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