DIRAC-HARMONIC MAPS FROM INDEX THEORY

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Abstract. We prove existence results for Dirac-harmonic maps using index theoretical tools. They are mainly interesting if the source manifold has dimension 1 or 2 modulo 8. Our solutions are uncoupled in the sense that the underlying map between the source and target manifolds is a harmonic map.

Contents

1. Introduction

Dirac-harmonic maps are stationary points of the fermionic analogue of the energy functional. The associated Euler-Lagrange equation is a coupled system consisting of a map $f : M \to N$ and a spinor on M twisted by f^*TN, such that the (twisted) spinor is in the kernel of the (twisted) Dirac operator. The goal of this article is to use index theoretical tools for providing such Dirac-harmonic maps.

1.1. **Dirac-harmonic maps.** In mathematical physics a (non-linear) sigma-model consists of Riemannian manifolds M and N. For simplicity we always assume M and N to be compact. The classical or bosonic energy of a map $f : M \to N$ is defined as $\frac{1}{2}\int_M |df|^2$. Stationary points of this functional are called harmonic maps. Harmonic maps from M to N are interpreted

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as solutions of this sigma-model. The term "non-linear" indicates that the target manifold N is not a vector space and will be omitted from now on.
A super-symmetric sigma-model (see [13, Chapter 3], [15]) also consists of Riemannian manifolds M and N, but in addition to above one assumes that M carries a fixed spin structure. A fermionic energy functional can now be defined on pairs (f,Φ), where f is again a map $M\to N$ and Φ is a spinor on M twisted by f^*TN. In the literature a linear and a non-linear versions of this functional are considered, see Subsection 3.2 for details. In our article and in most of the mathematical literature one considers the fermionic energy functional

$$L(f,\Phi)=\frac{1}{2}\int_M\Big(|df|^2+\langle\Phi,D\!\!\!\!/^f\Phi\rangle\Big)\,dv^M,$$

where $D\!\!\!\!/^f$ is the Dirac operator acting on spinors twisted by f^*TN. Stationary points (f_0,Φ_0) of this fermionic energy functional are called Dirac-harmonic maps. They are characterized by the equations $D\!\!\!\!/^{f_0}\Phi_0=0$ and $\mathrm{tr}_g(\nabla df_0)=\frac{V_{\Phi_0}}{2}$, see Proposition 5.1 below. They are interpreted as solutions of the super-symmetric sigma-model. Regularity and existence questions for Dirac-harmonic maps have been the subject of many articles recently where important results have been obtained, see Section 2.

1.2. **Trivial Dirac-harmonic maps.** A harmonic map together with a spinor that vanishes everywhere provides a Dirac-harmonic map. In the present article, such Dirac-harmonic maps are called *spinor-trivial* Dirac-harmonic maps.
Another class of obvious solutions consists of constant maps f together with a spinor Φ lying in the kernel of the (untwisted) Dirac operator, see Subsection 2.3. Such Dirac-harmonic maps will be called *map-trivial.* However, the dimension of the kernel depends in a subtle way on the conformal structure of M. For the sake of clarity we give a small review on known results about the kernel in Subsection 2.3.

1.3. **Main result.**

Definition 1.1. A Dirac-harmonic map will be called *uncoupled* if f is harmonic, otherwise it is called *coupled.*

Most of the existence results for Dirac-harmonic maps in the literature actually yield uncoupled Dirac-harmonic maps. Coupled Dirac-harmonic maps are discussed in [21, Thm. 3], see Subsection 2.2 for details.
Our result provides criteria for the existence of sufficiently many non-trivial, uncoupled Dirac-harmonic maps.

Theorem 1.2. *For M and N as above consider a homotopy class $[f]$ of maps $f:M^m\to N^n$ such that the $\mathrm{KO}_m(\mathrm{pt})$-valued index $\alpha(M,[f])$ is non-trivial. Assume that $f_0\in[f]$ is a harmonic map. Then there is a real* *vector space V of real dimension a_m such that all (f_0, Φ), $\Phi \in V$, are Dirac-harmonic maps, where*

$$a_m := \begin{cases} 2 & \text{if } m \equiv 0 \ (8) \\ 2 & \text{if } m \equiv 1 \ (8) \\ 4 & \text{if } m \equiv 2 \ (8) \\ 4 & \text{if } m \equiv 4 \ (8). \end{cases}$$

This theorem is an immediate consequence of Theorem 9.1. It is mainly interesting in dimensions 1 and 2 modulo 8. In dimensions divisible by 4 the statement directly follows from the Atiyah-Singer-index theorem and the grading techniques of Sections 6 and 7. In dimension 1 modulo 8, these grading techniques would also yield a similar result, but the space of Dirac-harmonic maps obtained this way would have only real dimension 1 instead of 2.

In order to derive such results we have structured the article as follows. We start by giving an overview about known results in Section 2 in order to compare them to our results. It is logically independent from the rest. Having introduced the notations and the fermionic energy in Section 3, we recall some well-known facts about the index of twisted Dirac operators in Section 4 and the variational formula in Section 5. This immediately leads to the first existence result using grading techniques (Section 6) and the result in dimensions m divisible by 4 (Section 7). In order to get better results in dimensions $8k+1$ and $8k+2$ we study the "minimal" and "non-minimal case" in Sections 8 and 9. Finally, we discuss in Section 10 numerous examples to which the above theorem applies.

2. Previously known results

To compare our results with the existing literature we briefly recall several known facts about Dirac-harmonic maps.

Results about regularity, investigations about the necessity for the mapping component f to be harmonic, removal of singularity theorems and blow-up analysis of Dirac-harmonic maps are developed in [9, 8, 38, 33, 39, 40]. We do not give details, as these issues are far from our topic.

We now summarize existence results for Dirac-harmonic maps. Not many concrete examples are known. Apart from the trivial examples in subsections 1.2 and 2.3, almost all examples occur in dimension $m = 2$ and can be divided into uncoupled and coupled solutions. Before we describe them, recall first that, in dimension $m = 2$, the existence of a Dirac-harmonic map only depends on the conformal class of the metric chosen on M: a pair (f_0, Φ_0) is Dirac-harmonic on (M^2, g) if and only if $(f_0, e^{-\frac{u}{2}}\Phi_0)$ is Dirac-harmonic on $(M^2, e^{2u}g)$, whatever $u \in C^\infty(M, \mathbb{R})$ is. From the point of view of string theory, the case $m = 2$ is of central importance as it describes the evolution of strings in space-times.

2.1. **Existence of uncoupled Dirac-harmonic maps.** The first existence result, which appears in [9, Prop. 2.2] for $M = N = \mathbb{S}^2$ and then in [21, Thm. 2] for general surfaces M, is based on an explicit construction involving a harmonic map and so-called *twistor-spinors* on the source manifold. Twistor-spinors are sections of the spinor bundle ΣM of M lying in the kernel of the Penrose operator $P : C^\infty(M, \Sigma M) \longrightarrow C^\infty(M, T^*M \otimes \Sigma M)$, $P_X\varphi := \nabla_X\varphi + \frac{1}{m}X \cdot \not{D}\varphi$ (here $\not{D}$ denotes the untwisted Dirac operator on M). The construction goes as follows. Let M be any m-dimensional (non necessarily compact) spin manifold and $f : M \longrightarrow N$ be a smooth map. For an arbitrary $\varphi \in C^\infty(M, \Sigma M)$ consider the smooth section $\Phi := \sum_{j=1}^m e_j \cdot \varphi \otimes df(e_j)$ of $\Sigma M \otimes f^*TN$, where $(e_j)_{1\leq j\leq m}$ is a local orthonormal basis of TM. Note that this (pointwise) definition does not depend on the choice of the local orthonormal basis $(e_j)_{1\leq j\leq m}$ and that the vanishing of Φ is equivalent to that of φ as soon as f is an immersion. Then a short computation shows that

$$\not{D}^f\Phi = -2\sum_{j=1}^m P_{e_j}\varphi \otimes df(e_j) + \frac{2-m}{m}\sum_{j=1}^m e_j \cdot \not{D}\varphi \otimes df(e_j) - \varphi \otimes \mathrm{tr}_g(\nabla df).$$

Therefore, if φ is a twistor-spinor, $m = 2$ and f is harmonic, then $\not{D}^f\Phi = 0$. Moreover, if $m = 2$, then one calculates that the vector field V_Φ defined in Proposition 5.1 vanishes. Therefore, (f, Φ) is a Dirac-harmonic map as soon as f is harmonic and the spinor φ involved in the definition of Φ is a twistor-spinor. This construction has the obvious drawback that the only closed surfaces admitting non-zero twistor-spinors are the 2-torus $\mathbb{T}^2$ with its non-bounding spin structure and the 2-sphere $\mathbb{S}^2$. In particular no example on a hyperbolic surface can be given using this approach. However, let us mention that all Dirac-harmonic maps from $\mathbb{S}^2$ to $\mathbb{S}^2$ are of that form [37, Thm. 1.2]. Vanishing results in case the spinor part Φ has the form above are considered in [26].

2.2. **Existence of coupled Dirac-harmonic maps.** Up to the knowledge of the authors, only two examples of coupled Dirac-harmonic maps are known. Both are constructed explicitly considering particular isometric but non-minimal immersions from special non-compact source manifolds into the hyperbolic space. The first one deals with an explicit non-minimal isoparametric immersion from a hyperbolic surface of revolution into the 3-dimensional hyperbolic space [21, Thm. 3].
The second one is provided by the totally umbilical (but non-totally geodesic) embedding of hyperbolic hyperplanes of sectional curvature $-\frac{4}{m+2}$ into the $m+1(\geq 4)$-dimensional hyperbolic space of sectional curvature -1. This example can be handled with the twistor-spinor-ansatz of [21] and is carried out in [3].

2.3. **More on trivial Dirac-harmonic maps.** In this subsection we will explain a product construction for producing examples of Dirac-harmonic maps, and then we will summarize the knowledge about map-trivial ones. The main objective of the article is to derive existence of Dirac-harmonic maps which do not fall in any of these trivial categories: spinor-trivial one, map-trivial ones, products of them.
To understand the product construction, assume that $f_1 : M \to N_1$ and $f_2 : M \to N_2$ are smooth maps, so that they are the components of the map $(f_1, f_2) : M \to N_1 \times N_2$. Similarly, let $\Phi_j \in \Gamma(\Sigma M \otimes f_j^*TN_j)$, $j = 1, 2$, and let $(\Phi_1, \Phi_2) \in \Gamma(\Sigma M \otimes (f_1^*TN_1 \oplus f_2^*TN_2)$ denote their sum. Let L (resp. L_i) be the fermionic energy functional for $N = N_1 \times N_2$ (resp. $N =$

N_i). Then $L((f_1, f_2), (\Phi_1, \Phi_2)) = L_1(f_1, \Phi_1) + L_2(f_2, \Phi_2)$. Thus, the map $((f_1, f_2), (\Phi_1, \Phi_2))$ is Dirac-harmonic if and only if both (f_1, Φ_1) and (f_2, Φ_2) are Dirac-harmonic.

So we obtain new examples of Dirac-harmonic maps by taking products of map-trivial Dirac harmonic maps with spinor-trivial maps.

The pair (f, Φ) is a map-trivial Dirac-harmonic map if and only if the following holds:

(1) f is constant, say $f \equiv x \in N$,
(2) $\Phi \in \Gamma(\Sigma M \otimes T_x N) \cong \Gamma(\Sigma M) \otimes \mathbb{R}^n$, $n = \dim N$, is a harmonic spinor, i.e. $\not{D}^f \Phi = 0$ and $\not{D}^f = \not{D} \otimes \mathrm{id}_{\mathbb{R}^n}$ where $\not{D} : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$ is the untwisted Dirac operator.

Thus, to determine the map-trivial Dirac-harmonic maps, one has to determine the kernel of the untwisted Dirac operator. The dimension of the kernel is invariant under conformal change of the metric, but it depends on the conformal class. Important progress was obtained in particular in [19], [4], [5] and [6]. The $\mathrm{KO}_m(\mathrm{pt})$-valued index obviously gives a lower bound on the dimension of the kernel. We say, that a Riemannian metric g is *$\not{D}$-minimal* if this lower bound is attained.

At first one knows that generic metrics are $\not{D}$-minimal.

Theorem 2.1 ([2])**.** *Let M be a connected compact spin manifold. Then the set of $\not{D}$-minimal Riemannian metrics is open and dense in the set of all Riemannian metrics in the C^k-topology for any $k \in \{1, 2, 3, \ldots\} \cup \{\infty\}$.*

The theorem was already proved in [6] for simply connected manifolds of dimension at least 5. There is also a stronger version of the theorem in [1] which states that any metric can be perturbed in an arbitrarily small neighborhood to a $\not{D}$-minimal one.

In the case $m = 2$ the complex dimension of the kernel of $\not{D}$ is at most $g+1$ where g is the genus of the surface M [19, Prop. 2.3]. This upper bound determines the dimension of the kernel of $\not{D}$ in the case $g \leq 2$, $\alpha(M) = 0$ and in the case $g \leq 4$, $\alpha(M) = 1$. The dimension of the kernel can also be explicitly determined for hyperelliptic surfaces [7]. In the same article the dimension of the kernel is calculated for special non-hyperelliptic surfaces of genus 4 and 6. However, for general surfaces of sufficiently large genus, the dimension of the kernel is unknown.

In contrast to $m = 2$, it is conjectured that for $m \geq 3$ there is no topological quantity that yields an upper bound on $\dim \ker \not{D}$.

Conjecture 2.2. [4, p. 941] *On any non-empty compact spin manifold of dimension $m \geq 3$, there is a sequence of Riemannian metrics g_i with*

$$\lim_{i \to \infty} \dim \ker \not{D}^{g_i} = \infty.$$

The conjecture is known only in few cases as e.g. the 3-sphere S^3 where Hitchin [19] has shown that a sequence of Berger metrics provides such a sequence of metrics.

In particular, the conjecture would include that any manifold with $m \geq 3$ carries a non-$\not{D}$-minimal metric. The latter statement is partially known to be true. It was shown in [19] for $m \equiv 7, 0, 1$ (8) and in [4] for $m \equiv$

$3,7$ (8) that any compact m-dimensional spin manifold carries a metric with $\ker D\!\!\!/ \neq 0$. A similar statement is known for all spheres of dimension $m \geq 4$ [31] and [12].
However it is still unknown whether an arbitrary compact spin manifold of dimension $m \equiv 2, 4, 5, 6$ (8) admits a non-$D\!\!\!/$-minimal metric.

3. Notation and Preliminaries

3.1. **Notation and Conventions.** The Riemannian manifolds M and N are always assumed to be compact without boundary, $m = \dim M$, $n = \dim N$. We assume that M is spin and — unless stated otherwise — we always assume that one of the spin structures on M is fixed. The (untwisted) spinor bundle on M is denoted by ΣM, the fiber over $x \in M$ is denoted by $\Sigma_x M$. In this article we consider complex spinors, i. e. , ΣM is a complex vector bundle of rank $2^{[\frac{m}{2}]}$; similar statements also hold for real spinors. The untwisted spinor bundle carries a Hermitian metric, a metric connection and a Clifford multiplication. For an introduction to these structures we refer to textbooks on spin geometry such as [23, 16, 18]. These structures allow to define an (untwisted) Dirac operator $D\!\!\!/ : C^\infty(M, \Sigma M) \longrightarrow C^\infty(M, \Sigma M)$.
If E is a real vector bundle with given metric and metric connection, we define the twisted spinor bundle as $\Sigma M \otimes E$, the tensor bundle being taken over $\mathbb{R}$. The twisted spinor bundle carries similar structures as the untwisted one. Sections of $\Sigma M \otimes E$ will be called *spinors* or *spinors twisted by* E. One uses the structures on $\Sigma M \otimes E$ to define the twisted Dirac operator

$$D\!\!\!/^E : C^\infty(M, \Sigma M \otimes E) \longrightarrow C^\infty(M, \Sigma M \otimes E).$$

It is a first order elliptic and selfadjoint differential operator. Its spectrum is thus discrete, real and of finite multiplicity.
In this article E will often be obtained as $E = f^*TN$ where $f : M \to N$ is smooth. Here E is equipped with the pull-back of the metric and the Levi-Civita connection on TN. In this case we simply write $D\!\!\!/^f$ for $D\!\!\!/^{f^*TN}$.
Our convention for curvature tensors is $R^N_{X,Y} = [\nabla^N_X, \nabla^N_Y] - \nabla^N_{[X,Y]}$, and R is then considered as an element in $\mathrm{Hom}(TN \otimes TN \otimes TN, TN)$ through $R(X \otimes Y \otimes Z) = R(X, Y)Z$.
The sphere S^n with its standard metric (of constant sectional curvature 1) will be denoted as $\mathbb{S}^n$.
The complex projective space $\mathbb{C}\mathrm{P}^n$ will be always equipped with its Fubini-Study metric of holomorphic sectional curvature 4. The tautological bundle on $\mathbb{C}\mathrm{P}^n$ is denoted by γ_n.

3.2. **Linear and non-linear energy functional.** As mentioned in the introduction, the fermionic energy functional is discussed in a linear and non-linear version in the literature. We will give some details now. For a concise introduction see [15] or [13, Chapter 3].
The most appropriate functional from the physical point of view would be Grassmann-algebra-valued. However, as this formalism is a bit overloaded for analytical purposes, one usually works with simplified models. One often

considers the following fermionic energy functional

$$L_c(f,\Phi) := \frac{1}{2}\int_M (|df|^2 + \langle\Phi, D\!\!\!/^f\Phi\rangle + \frac{1}{6}\langle\Phi, \mathcal{R}(\Phi,\Phi)\Phi\rangle)\, dv^M.$$

Here $\mathcal{R}$ is a section of $\mathrm{Hom}_{\mathbb{C}}((\Sigma M \otimes_{\mathbb{R}} TN) \otimes_{\mathbb{C}} (\overline{\Sigma M} \otimes_{\mathbb{R}} TN) \otimes_{\mathbb{C}} (\Sigma M \otimes_{\mathbb{R}} TN), \Sigma M \otimes_{\mathbb{R}} TN)$ obtained by tensoring the Riemann curvature tensor $R \in \mathrm{Hom}_{\mathbb{R}}(TN \otimes_{\mathbb{R}} TN \otimes_{\mathbb{R}} TN, TN)$ with the spinorial contraction map $\mathrm{id} \otimes \langle\,.\,\rangle \in \mathrm{Hom}_{\mathbb{C}}(\Sigma M \otimes_{\mathbb{C}} \overline{\Sigma M} \otimes_{\mathbb{C}} \Sigma M, \Sigma M)$, $\mathrm{id} \otimes \langle\,.\,\rangle(\varphi_1,\varphi_2,\varphi_3) = \langle\varphi_2,\varphi_3\rangle\varphi_1$. This functional is often called the fermionic energy functional with curvature term, and stationary points of this functional are called *Dirac-harmonic maps with curvature term*. However this functional makes analytical considerations involved, and thus only few analytical articles include this curvature term, e. g. [11] and [10]. In [20] interesting techniques are developed which might turn helpful to find solutions of the non-linear equation.

As the curvature term is of fourth order in Φ whereas the dominating term is quadratic on Φ, it seems acceptable from the view point of physical applications to neglect the curvature term. The fermionic energy functional thus obtained

$$L(f,\Phi) := \frac{1}{2}\int_M (|df|^2 + \langle\Phi, D\!\!\!/^f\Phi\rangle)\, dv^M \in \mathbb{R},$$

is analytically much easier to study. A pair (f_0, Φ_0) is a stationary point of L if and only if it satisfies $D\!\!\!/^{f_0}\Phi_0 = 0$ and $\mathrm{tr}_g(\nabla df_0) = \frac{V_{\Phi_0}}{2}$, see Proposition 5.1 below. Stationary points of this functional are called *Dirac-harmonic maps*. Starting with [9], this functional has been intensively studied in the literature, see Subsection 2.

The method developed in the present article applies to this linear version.

4. Index theory of Dirac-harmonic maps

Let M be a closed m-dimensional Riemannian spin manifold with spin structure denoted by χ. Let $E \longrightarrow M$ be a Riemannian (real) vector bundle with metric connection. Then one can associate to the twisted Dirac operator $D\!\!\!/^E : C^\infty(M, \Sigma M \otimes E) \longrightarrow C^\infty(M, \Sigma M \otimes E)$ an index $\alpha(M,\chi,E) \in \mathrm{KO}_m(\mathrm{pt})$ (see e. g. [23, p.151]). Using the isomorphism [23, p.141]

$$\mathrm{KO}_m(\mathrm{pt}) \cong \begin{cases} \mathbb{Z} & \text{if } m \equiv 0\ (4) \\ \mathbb{Z}_2 & \text{if } m \equiv 1,2\ (8) \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

the index $\alpha(M,\chi,E)$ will be identified either with an integer or an element in the group $\mathbb{Z}_2$ of integers modulo 2. We also say that $\alpha(M,\chi,E)$ is the *α-genus* of $E \longrightarrow M$.

The α-genus can be easily determined out of $\ker D\!\!\!/^E$ using the following formula [23, Thm. II.7.13]:

$$\alpha(M,\chi,E) = \begin{cases} \{\mathrm{ch}(E)\cdot\widehat{A}(TM)\}[M] & \text{if } m \equiv 0\ (8). \\ \left[\dim_{\mathbb{C}}(\ker(D\!\!\!/^E))\right]_{\mathbb{Z}_2} & \text{if } m \equiv 1\ (8) \\ [\frac{\dim_{\mathbb{C}}(\ker(D\!\!\!/^E))}{2}]_{\mathbb{Z}_2} & \text{if } m \equiv 2\ (8) \\ \frac{1}{2}\{\mathrm{ch}(E)\cdot\widehat{A}(TM)\}[M] & \text{if } m \equiv 4\ (8) \end{cases}$$

As usual $\mathrm{ch}(E)$ is to be understood as the Chern character of $E \otimes_{\mathbb{R}} \mathbb{C}$. In dimensions 1 and 2 modulo 8 the α-genus depends on the spin structure χ on M, however we often simply write $\alpha(M, E)$, when χ is clear from the context. We will assume from now on and until the end of Section 9 that M comes with a fixed orientation and spin structure, so we omit the notation χ in those sections.
The number $\alpha(M, E)$ is a spin-bordism-invariant, where a spin-bordism for manifold with vector bundles means that the restriction of a vector bundle to the boundary of the spin manifold must coincide with the vector bundle given on the boundary: namely, [23, Thm. II.7.14] implies that α defines a map $\Omega_*^{\mathrm{Spin}}(\mathrm{BO}) \longrightarrow \mathrm{KO}_*(\mathrm{pt})$. Note in particular that $\alpha(M, E)$ does depend neither on the metric nor on the connection chosen on $E \longrightarrow M$. In case that E is the trivial real line bundle $\underline{\mathbb{R}} := \mathbb{R} \times M \longrightarrow M$, the number $\alpha(M, E) =: \alpha(M)$ is the classical α-genus of M.

Definition 4.1. Let M be a closed m-dimensional Riemannian spin manifold and $f : M \to N$ be a smooth map into an n-dimensional Riemannian manifold. The *α-genus* of f is $\alpha(M, f) := \alpha(M, f^*TN)$.

The spin-bordism-invariance of the α-genus has an important consequence for $\alpha(M, f)$. We first make a

Definition 4.2. With the notations of Definition 4.1, two maps $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are *spin-bordant* in N, if there is a spin manifold W of dimension $m+1$ together with a map $F : W \to N$ such that $\partial W = -M_1 \amalg M_2$ (in the sense of manifolds with spin structure) and such that $F_{|M_j} = f_j$ for both $j = 1, 2$.

Obviously, given any smooth spin-bordant maps $f_j : M_j \longrightarrow N$, $j = 1, 2$, the pull-back bundles $f_j^*(TN) \longrightarrow M_j$ are spin-bordant as vector bundles. Therefore, we obtain the

Proposition 4.3. *Assume that $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are spin-bordant in N. Then $\alpha(M_1, f_1) = \alpha(M_2, f_2)$.*

Since a homotopy between maps $f_j : M \longrightarrow N$ defines a spin-bordism between them, we deduce the

Corollary 4.4. *Assume that $f_1 : M \to N$ and $f_2 : M \to N$ are homotopic maps. Then $\alpha(M, f_1) = \alpha(M, f_2)$.*

Because of this homotopy invariance we also write $\alpha(M, [f])$ for $\alpha(M, f)$, where $[f]$ is the homotopy class of f.

5. Variational formulas for the fermionic energy functional

In the following, we denote by h the Riemannian metric on N. If $f : M \to N$ is a smooth map, then M carries the twisted spinor bundle $\Sigma M \otimes f^*TN$. For given $x \in M$ and vectors $X \in T_xM$ and $Y \in T_{f(x)}N$, we denote by $X \cdot \otimes R^N_{Y,df(X)}$ the linear endomorphism obtained by tensorizing the Clifford multiplication by X (acting on $\Sigma_x M$) with the curvature $R^N_{Y,df(X)} \in$

$\mathrm{End}(T_{f(x)}N)$. It is the real tensor product of a skew-hermitian endomorphism with a skew-symmetric endomorphism, and thus a hermitian endomorphism. In particular, the expression $\langle X \cdot \otimes R^N_{Y,df(X)}\Phi_0, \Phi_0\rangle$ is real. It is easy to notice that, if a local orthonormal basis $(e_j)_{1\leq j\leq m}$ of TM is chosen, then the pointwise (hermitian) endomorphism $\sum_{j=1}^m e_j \cdot \otimes R^N_{Y,df_0(e_j)}$ of $\Sigma M \otimes f^*TN$ does not depend on the choice of the basis $(e_j)_{1\leq j\leq m}$ and hence defines a global section of $\mathrm{End}(\Sigma M \otimes f^*TN)$. We recall the following formulas for the first variation of L [9, Prop. 2.1].

Proposition 5.1. *For $(f_0, \Phi_0) \in C^\infty(M,N) \times C^\infty(M, \Sigma M \otimes f_0^*TN)$ and some $\varepsilon > 0$ let $(f_t, \Phi_t)_{t\in]-\varepsilon,\varepsilon[}$ be a variation of (f_0,Φ_0) which is differentiable at $t = 0$. Denote by $D\!\!\!\!/\,_t := D\!\!\!\!/\,^{f_t}$ for all $t \in]-\varepsilon, \varepsilon[$. Then*

$$\frac{d}{dt}\langle D\!\!\!\!/\,_t\Phi_t, \Phi_t\rangle_{|t=0} = h(V_{\Phi_0}, \frac{\partial f}{\partial t}(0)) + \langle D\!\!\!\!/\,_0 \frac{\partial \Phi}{\partial t}(0), \Phi_0\rangle + \langle D\!\!\!\!/\,_0\Phi_0, \frac{\partial \Phi}{\partial t}(0)\rangle \quad (2)$$

where the section V_{Φ_0} of f_0^*TN is defined by the pointwise relation $h(V_{\Phi_0}, Y) = \sum_{j=1}^m \langle e_j \cdot \otimes R^N_{Y,df_0(e_j)}\Phi_0, \Phi_0\rangle$ for all $Y \in TN$ and every choice of local orthonormal frame $(e_j)_{1\leq j\leq m}$ of TM. In particular,

$$\frac{d}{dt}L(f_t,\Phi_t)_{|t=0} = \int_M \Re e(\langle D\!\!\!\!/\,_0\Phi_0, \frac{\partial \Phi}{\partial t}(0)\rangle) - h\left(\mathrm{tr}_g(\nabla df_0) - \frac{V_{\Phi_0}}{2}, \frac{\partial f}{\partial t}(0)\right) dv^M.$$

The differentiability of $t \mapsto \Phi_t$ at $t = 0$ is to be understood as follows. Denote by $f :]-\varepsilon,\varepsilon[\times M \longrightarrow N$, $(t,x) \mapsto f(t,x) := f_t(x)$ the variation above. The metric and the Levi-Civita connection on TN induce a metric and a metric connection on the pull-back bundle $f^*TN \longrightarrow]-\varepsilon,\varepsilon[\times M$. For every $t \in]-\varepsilon,\varepsilon[$ let $\beta_t : C^\infty(M, \Sigma M \otimes f_0^*TN) \longrightarrow C^\infty(M, \Sigma M \otimes f_t^*TN)$ be the unitary and parallel isomorphism induced by the parallel transport along the curves $[0,t] \to]-\varepsilon,\varepsilon[\times M$, $s \mapsto (s,x)$, where x runs in M. Then we require the map $t \mapsto \widehat{\Phi_t} := \beta_t^{-1} \circ \Phi_t \in C^\infty(M, \Sigma M \otimes f_0^*TN)$ to be differentiable at $t = 0$ in the following sense: the map $]-\varepsilon,\varepsilon[\longrightarrow C^\infty(M, \Sigma M \otimes f_0^*TN)$, $t \mapsto \widehat{\Phi}_t$, has a derivative at $t = 0$ which is at least continuous on M. Here we consider the topology induced by the $\mathrm{H}^{1,2}$-norm on $C^\infty(M, \Sigma M \otimes f_0^*TN)$. In that case, we denote by $\frac{\partial \Phi}{\partial t}(0) := \frac{\partial \widehat{\Phi}}{\partial t}(0) \in C^1(M, \Sigma M \otimes f_0^*TN)$. Note that, if $\widehat{D\!\!\!\!/}_t := \beta_t^{-1} \circ D\!\!\!\!/\,_t \circ \beta_t : C^\infty(M, \Sigma M \otimes f_0^*TN) \longrightarrow C^\infty(M, \Sigma M \otimes f_0^*TN)$, then $\langle \widehat{D\!\!\!\!/}_t\widehat{\Phi}_t, \widehat{\Phi}_t\rangle = \langle D\!\!\!\!/\,_t\Phi_t, \Phi_t\rangle$ since β_t is unitary.

As a straightforward consequence of Proposition 5.1, we obtain:

Corollary 5.2. *Let $f_0 \in C^\infty(M,N)$ be a harmonic map and $\Phi_0 \in \ker(D\!\!\!\!/\,^{f_0})$. Assume the existence, for every smooth variation $(f_t)_{t\in]-\varepsilon,\varepsilon[}$ of f_0, of a variation $(\Phi_t)_{t\in]-\varepsilon,\varepsilon[}$ of Φ_0 which is differentiable at $t = 0$ and such that $\frac{d}{dt}\left(D\!\!\!\!/\,^{f_t}\Phi_t, \Phi_t\right)_{\mathrm{L}^2}{}_{|t=0} = 0$. Then the pair (f_0, Φ_0) is an uncoupled Dirac-harmonic map.*

Proof. Only $V_{\Phi_0} = 0$ has to be proved, which follows from integrating (2) and using the self-adjointness of $D\!\!\!\!/\,^{f_0}$. $\square$

Corollary 5.2 is the corner stone for the proof of our main result, because it implies the following: to obtain the pair (f_0, Φ_0) (with f_0 harmonic) as a

Dirac-harmonic map, it suffices to construct, for any smooth variation $(f_t)_t$ of f_0, a sufficiently smooth one-parameter family of twisted spinors $(\Phi_t)_t$ with $\left(D\!\!\!/^{f_t}\Phi_t, \Phi_t\right)_{\mathrm{L}^2} = 0$ for all sufficiently small t. And this can be achieved either with considerations about the $\mathbb{Z}_2$-grading of the twisted spinor bundle (see Corollary 6.1) or with a non-trivial α-genus (see Proposition 8.2).

6. The graded case

Let $f \in C^\infty(M,N)$ be any map. In this section we consider the situation where the map bundle ΣM admits an orthogonal and parallel $\mathbb{Z}_2$-grading $G \in \mathrm{End}(\Sigma M)$, $G^2 = \mathrm{id}$, anti-commuting with the Clifford-multiplication by tangent vectors. Such a grading naturally exists in even dimensions and in dimension $m \equiv 1$ (8) and induces a grading of $\Sigma M \otimes f^*TN$.
In even dimension m, it is given by the Clifford action of the so-called complex volume form of TM, namely $G(\varphi) = i^{\frac{m}{2}} e_1 \cdot \ldots \cdot e_m \cdot \varphi$ for a positively oriented orthonormal local frame. One easily checks that the definition of G does not depend on the choice of the local frame, and thus G is globally defined. The spinor bundle then decomposes into two complex subbundles $\Sigma_+ M$ and $\Sigma_- M$ associated to the $+1$ and -1-eigenvalue of G respectively. As G is Hermitian and parallel, the decomposition $\Sigma M = \Sigma_+ M \oplus \Sigma_- M$ is orthogonal in the complex sense and parallel.
In dimension $m \equiv 1$ (8), the grading G is provided by a real structure on the complex spinor representation, see e. g. [16, Sec. 1.7]. This map is complex anti-linear, involutive and anticommutes with the Clifford action by tangent vectors. It is still self-adjoint in the real sense, i. e. $\Re e(\langle G(\varphi), \psi\rangle) = \Re e(\langle \varphi, G(\psi)\rangle)$ for all spinors φ, ψ. Thus the real structure induces a real-orthogonal and parallel decomposition $\Sigma M = \Sigma_+ M \oplus \Sigma_- M$ into *real* subbundles associated to the eigenvalues ± 1. We also have $\Sigma M = \Sigma_+ M \otimes_{\mathbb{R}} \mathbb{C}$.

Since f^*TN is considered as a real vector bundle, we obtain a $\mathbb{Z}_2$-grading $G \otimes \mathrm{id}$ on the tensor product $\Sigma M \otimes f^*TN$ anticommuting with the Clifford multiplication by tangent vectors. In particular $G \otimes \mathrm{id}$ anti-commutes with $D\!\!\!/^f$ which hence splits into $D\!\!\!/^f_+ + D\!\!\!/^f_-$, where

$$D\!\!\!/^f_\pm : C^\infty(M, \Sigma_\pm M \otimes f^*TN) \longrightarrow C^\infty(M, \Sigma_\mp M \otimes f^*TN).$$

From the orthogonality of the splitting, we see that $\Re e(\langle D\!\!\!/^f \Phi_+, \Phi_+\rangle) = \Re e(\langle D\!\!\!/^f \Phi_-, \Phi_-\rangle) = 0$ for all $\Phi_\pm \in C^\infty(M, \Sigma_\pm M \otimes f^*TN)$. On the other hand, $(D\!\!\!/^f \Phi_\pm, \Phi_\pm)_{\mathrm{L}^2}$ is real as $D\!\!\!/^f$ is self-adjoint. Thus $(D\!\!\!/^f \Phi_+, \Phi_+)_{\mathrm{L}^2} = (D\!\!\!/^f \Phi_-, \Phi_-)_{\mathrm{L}^2} = 0$. Therefore, Corollary 5.2 implies:

Corollary 6.1. *Assume m is even or $m \equiv 1$ (8). Let $f_0 \in C^\infty(M^m, N)$ be a harmonic map. Split $\Phi_0 \in \ker(D\!\!\!/^{f_0})$ into $\Phi_0 = \Phi_0^+ + \Phi_0^-$. Then the pairs (f_0, Φ_0^+) and (f_0, Φ_0^-) are Dirac-harmonic.*

Remark 6.2. An alternative proof of Corollary 6.1 is obtained by showing $V_{G\otimes\mathrm{id}(\Phi_0)} = -V_{\Phi_0}$ for all spinors φ_0. If $\Phi_0 = \pm G \otimes \mathrm{id}(\Phi_0)$, then using $V_{-\Phi_0} = V_{\Phi_0}$, we obtain $V_{\Phi_0} = 0$ and hence (f_0, Φ_0) is Dirac-harmonic.

Corollary 6.1 is of central importance, since it will shows the existence of non-trivial Dirac-harmonic maps in the case where the underlying map f_0 is not perturbation-minimal, see Section 9.

7. The case $m \in 4\mathbb{N}$

As noticed above, if we assume m to be even, the Dirac operator D^f is odd with respect to the above grading. Restriction to sections of $\Sigma_+ M$ thus yields an operator

$$\not{D}^f_+ : C^\infty(M, \Sigma_+ M) \to C^\infty(M, \Sigma_- M).$$

It is a Fredholm operator and the Atiyah-Singer index theorem yields its (complex) index:

$$\mathrm{ind}(\not{D}^f_+) = \{\mathrm{ch}(f^*TN) \cdot \widehat{A}(TM)\}[M].$$

This implies

$$\mathrm{ind}(\not{D}^f_+) = \dim_{\mathbb{C}} \ker(\not{D}^f_+) - \dim_{\mathbb{C}} \ker(\not{D}^f_-) = \begin{cases} \alpha(M, f) & \text{if } m \equiv 0 \quad (8) \\ 0 & \text{if } m \equiv 2, 6 \quad (8) \\ 2\,\alpha(M, f) & \text{if } m \equiv 4 \quad (8) \end{cases}$$

This follows from the definition of α if m is a multiple of 4 and from the fact that the real (case $m \equiv 6$ (8)) or quaternionic (case $m \equiv 2$ (8)) structure on ΣM exchanges $\ker \not{D}^f_+$ and $\ker \not{D}^f_-$ if $m \equiv 2, 6$ (8). Alternatively, the statement in the case $m \equiv 6$ (8) can be deduced from the index theorem.
We now restrict to the case that m is a multiple of 4. Corollary 6.1 yields the following

Corollary 7.1. *Let $m \in 4\mathbb{N}$. Assume $f_0 \in C^\infty(M^m, N)$ to be a harmonic map with $\mathrm{ind}(\not{D}^{f_0}_+) \neq 0$. Set $\varepsilon := \mathrm{sign}(\mathrm{ind}(\not{D}^{f_0}_+)) \in \{+, -\}$. Then*

$$\{(f_0, \Phi_0^\varepsilon) \,|\, \Phi_0^\varepsilon \in \ker(\not{D}^{f_0}_\varepsilon)\}$$

is a linear space of Dirac-harmonic maps of complex dimension at least $|\mathrm{ind}(\not{D}^{f_0}_+)|$. This complex dimension is even for $m \equiv 4$ (8).

8. Minimality

The present and the following sections provide results for any dimension m, but are mainly interesting if $m \equiv 1, 2$ (8).

Definition 8.1. A smooth map $f_0 : M \longrightarrow N$ is called *perturbation-minimal* if and only if $\dim(\ker(\not{D}^{f^*TN})) \geq \dim(\ker(\not{D}^{f_0^*TN}))$ for all f in a C^∞-neighbourhood of f_0.

Obviously, any homotopy class of maps from M to N contains perturbation-minimal maps.

Proposition 8.2. *Let $f_0 : M \to N$ be perturbation-minimal and harmonic. Then for any $\Phi_0 \in \ker(\not{D}^{f_0})$, the pair (f_0, Φ_0) is an uncoupled Dirac-harmonic map.*

Proof. Since f_0 is harmonic and $D\!\!\!/^{f_0}\Phi_0 = 0$, it suffices by Corollary 5.2 to show that, for every smooth variation $(f_t)_{t\in]-\varepsilon,\varepsilon[}$ of f_0, there exists a variation $(\Phi_t)_t$ of Φ_0, differentiable at $t=0$, such that $D\!\!\!/^{f_t}\Phi_t = 0$ for all t close enough to 0. To that extent we fix such a variation $(f_t)_{t\in]-\varepsilon,\varepsilon[}$ of f_0 and consider, for each $t\in]-\varepsilon,\varepsilon[$, the L^2-orthogonal projection $\pi_t : \mathrm{L}^2(M,\Sigma M\otimes f_0^*(TN)) \longrightarrow \ker(\widehat{D\!\!\!/}_t) \subset C^\infty(M,\Sigma M\otimes f_0^*TN)$, where $\widehat{D\!\!\!/}_t := \beta_t^{-1}\circ D\!\!\!/_t\circ\beta_t$ and $D\!\!\!/_t := D\!\!\!/^{f_t}$, see Proposition 5.1.

Claim A: *For every $\Psi_0\in\ker(D\!\!\!/_0)$ one has $\|\pi_t(\Psi_0)-\Psi_0\|_{\mathrm{L}^2} = \mathrm{O}(t)$ as $t\to 0$.*

Proof of Claim A. By assumption, one has $\dim(\ker(\widehat{D\!\!\!/}_t)) \geq \dim(\ker(\widehat{D\!\!\!/}_0)) = \dim(\ker(D\!\!\!/_0))$ for all t small enough. Since the dimension of the kernel is always upper semi-continuous in the parameter, this already implies the equality $\dim(\ker(\widehat{D\!\!\!/}_t)) = \dim(\ker(D\!\!\!/_0)) =: k$ for all t in a sufficiently small neighbourhood of 0. Let now $\{\lambda_j(\widehat{D\!\!\!/}_t)\}_{j=1}^\infty$ denote the spectrum of $\widehat{D\!\!\!/}_t$, where $0\leq|\lambda_1|\leq\ldots\leq|\lambda_j|\leq|\lambda_{j+1}|\leq\ldots$. Since the map $t\mapsto f_t$ is smooth, the spectrum of $\widehat{D\!\!\!/}_t$ depends continuously on t (each eigenvalue is continuous in t) [34, Sec. 9.3], in particular $t\mapsto\lambda_{k+1}(\widehat{D\!\!\!/}_t)$ is continuous. But the condition $\dim(\ker(\widehat{D\!\!\!/}_t)) = k$ forces $\lambda_{k+1}(\widehat{D\!\!\!/}_t)$ to be positive for all small enough t, therefore there exists $\lambda_0>0$ such that $|\lambda_{k+1}(\widehat{D\!\!\!/}_t)|\geq\lambda_0$ for all small t. The min-max principle then implies that, for every $\Psi\in\ker(\widehat{D\!\!\!/}_t)^\perp\cap\mathrm{H}^{1,2}(M)$,

$$\|\widehat{D\!\!\!/}_t\Psi\|_{\mathrm{L}^2}\geq|\lambda_{k+1}(\widehat{D\!\!\!/}_t)|\cdot\|\Psi\|_{\mathrm{L}^2}\geq\lambda_0\cdot\|\Psi\|_{\mathrm{L}^2}.$$

Putting $\Psi=\pi_t(\Psi_0)-\Psi_0\in\ker(\pi_t)\cap\mathrm{H}^{1,2}(M)=\ker(\widehat{D\!\!\!/}_t)^\perp\cap\mathrm{H}^{1,2}(M)$, we obtain

$$\begin{aligned}\lambda_0\cdot\|\pi_t(\Psi_0)-\Psi_0\|_{\mathrm{L}^2} &\leq \|\widehat{D\!\!\!/}_t(\pi_t(\Psi_0)-\Psi_0)\|_{\mathrm{L}^2}\\ &= \|(\widehat{D\!\!\!/}_0-\widehat{D\!\!\!/}_t)(\Psi_0)\|_{\mathrm{L}^2}\\ &\leq \|\widehat{D\!\!\!/}_0-\widehat{D\!\!\!/}_t\|_{\mathrm{op}}\cdot\|\Psi_0\|_{\mathrm{H}^{1,2}},\end{aligned}$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm for bounded linear operators from $\mathrm{H}^{1,2}(M)$ into $\mathrm{L}^2(M)$. Since by construction all operators $\widehat{D\!\!\!/}_t$ have the same principal symbol and depend smoothly on the parameter t, it is easy to see that $\|\widehat{D\!\!\!/}_0-\widehat{D\!\!\!/}_t\|_{\mathrm{op}}=\mathrm{O}(t)$ and hence $\|\pi_t(\Psi_0)-\Psi_0\|_{\mathrm{L}^2}=\mathrm{O}(t)$ as t goes to 0, which was to be shown. ✓

Claim B: *For every $\Psi_0\in\ker(D\!\!\!/_0)$, the limit $\lim\limits_{t\to0}\frac{\pi_t(\Psi_0)-\Psi_0}{t}$ exists in* $\mathrm{H}^{1,2}$ *and is actually smooth on M.*

Proof of Claim B. For obvious reasons (see (2) above), we use the short notation $\frac{d}{dt}\widehat{D\!\!\!/}_{t|_{t=0}} := \sum_{j=1}^m e_j\cdot\otimes R^N_{\frac{\partial f}{\partial t}(0),df_0(e_j)}\in C^\infty(M,\mathrm{End}(\Sigma M\otimes f_0^*TN))$. We first show that $\frac{d}{dt}\widehat{D\!\!\!/}_{t|_{t=0}}\Psi_0\in\ker(D\!\!\!/_0)^\perp\cap\mathrm{H}^{1,2}(M)$ (as in Claim A, the orthogonal complement is considered in L^2). Let $X_0\in\ker(D\!\!\!/_0)$, then for every small $t\neq0$ one has $(\frac{d}{dt}\widehat{D\!\!\!/}_{t|_{t=0}}\Psi_0,X_0)_{\mathrm{L}^2} = \lim\limits_{t\to0}(\frac{1}{t}\widehat{D\!\!\!/}_t\Psi_0,X_0)_{\mathrm{L}^2}$, where the convergence of the zero-order-operator $\frac{D\!\!\!/_t-D\!\!\!/_0}{t}$ to $\frac{d}{dt}\widehat{D\!\!\!/}_{t|_{t=0}}$ is to be understood in the C^0-sense (and not in the operator norm $\|\cdot\|_{\mathrm{op}}$ above). Now

$$\begin{aligned}(\frac{1}{t}\widehat{D\!\!\!/}_t\Psi_0, X_0)_{\mathrm{L}^2} &= (\frac{1}{t}\cdot\Psi_0, \widehat{D\!\!\!/}_t X_0)_{\mathrm{L}^2}\\ &= (\Psi_0-\pi_t(\Psi_0), \frac{1}{t}\widehat{D\!\!\!/}_t X_0)_{\mathrm{L}^2}\end{aligned}$$

because of $\widehat{D\!\!\!/}_t(\pi_t(\Psi_0)) = 0$. But $\frac{1}{t}\widehat{D\!\!\!/}_t X_0 \underset{t\to 0}{\longrightarrow} \frac{d}{dt}\widehat{D\!\!\!/}_{t|_{t=0}} X_0$ (which is well-defined and lies in $C^\infty(M, \Sigma M\otimes f_0^*TN)$) and $\Psi_0 - \pi_t(\Psi_0) \underset{t\to 0}{\longrightarrow} 0$ in L^2 by Claim A, therefore $(\frac{1}{t}\widehat{D\!\!\!/}_t\Psi_0, X_0)_{\mathrm{L}^2} \underset{t\to 0}{\longrightarrow} 0$, and this implies

$$(\frac{d}{dt}\widehat{D\!\!\!/}_{t|_{t=0}}\Psi_0, X_0)_{\mathrm{L}^2} = 0.$$

Since this holds for every $X_0\in\ker(D\!\!\!/_0)$, we obtain $\frac{d}{dt}\widehat{D\!\!\!/}_{t|_{t=0}}\Psi_0 \in \ker(D\!\!\!/_0)^\perp$. Elliptic regularity yields the smoothness of Ψ_0 and thus $\frac{d}{dt}\widehat{D\!\!\!/}_{t|_{t=0}}\Psi_0$ is smooth as well.

Hence there exists a unique $\Theta_0\in\ker(D\!\!\!/_0)^\perp$ such that $D\!\!\!/_0\Theta_0 = -\frac{d}{dt}\widehat{D\!\!\!/}_{t|_{t=0}}\Psi_0$. Note that, because of the ellipticity of $D\!\!\!/_0$, the section Θ_0 has to be smooth. We show that $\lim\limits_{t\to 0}\frac{\pi_t(\Psi_0)-\Psi_0}{t} = \Theta_0$ in $\mathrm{H}^{1,2}$. By Claim A, we have, for every $t\neq 0$,

$$\begin{aligned}D\!\!\!/_0(\frac{\pi_t(\Psi_0)-\Psi_0}{t} - \Theta_0) &= \frac{(D\!\!\!/_0 - D\!\!\!/_t)}{t}(\pi_t(\Psi_0)) - D\!\!\!/_0\Theta_0\\ &\underset{t\to 0}{\overset{\mathrm{L}^2}{\longrightarrow}} -\frac{d}{dt}\widehat{D\!\!\!/}_{t|_{t=0}}\Psi_0 + \frac{d}{dt}\widehat{D\!\!\!/}_{t|_{t=0}}\Psi_0,\end{aligned}$$

that is, $\|D\!\!\!/_0(\frac{\pi_t(\Psi_0)-\Psi_0}{t} - \Theta_0)\|_{\mathrm{L}^2} \underset{t\to 0}{\longrightarrow} 0$. Since $D\!\!\!/_0$ is elliptic, it remains to show that $\|\frac{\pi_t(\Psi_0)-\Psi_0}{t} - \Theta_0\|_{\mathrm{L}^2} \underset{t\to 0}{\longrightarrow} 0$. Pick any $X_0\in\ker(D\!\!\!/_0)$, then for any $t\neq 0$,

$$\begin{aligned}\big(\frac{\pi_t(\Psi_0)-\Psi_0}{t} - \Theta_0, X_0\big)_{\mathrm{L}^2} &= \big(\frac{\pi_t(\Psi_0)-\Psi_0}{t}, X_0\big)_{\mathrm{L}^2}\\ &= \big(\frac{\pi_t(\Psi_0)-\Psi_0}{t}, X_0 - \pi_t(X_0)\big)_{\mathrm{L}^2}.\end{aligned}$$

Since by Claim A both $\frac{\pi_t(\Psi_0)-\Psi_0}{t}$ remains L^2-bounded near $t=0$ and $X_0 - \pi_t(X_0) \underset{t\to 0}{\longrightarrow} 0$ in L^2, we deduce that $\big(\frac{\pi_t(\Psi_0)-\Psi_0}{t} - \Theta_0, X_0\big)_{\mathrm{L}^2} \underset{t\to 0}{\longrightarrow} 0$. This holds for any X_0 in the finite-dimensional space $\ker(D\!\!\!/_0)$, therefore $\|\frac{\pi_t(\Psi_0)-\Psi_0}{t} - \Theta_0\|_{\mathrm{L}^2} \underset{t\to 0}{\longrightarrow} 0$, which yields the result. ✓

It follows from Claim B that, setting $\widehat{\Phi}_t := \pi_t(\Phi_0)$, then $\widehat{\Phi}_t\in C^\infty(M, \Sigma M\otimes f_0^*TN)$ is a solution of $\widehat{D\!\!\!/}_t\widehat{\Phi}_t = 0$ for every t with $\widehat{\Phi}_{t|_{t=0}} = \Phi_0$ and the map $]-\varepsilon,\varepsilon[\longrightarrow \mathrm{H}^{1,2}(M)$, $t\mapsto\widehat{\Phi}_t$ is differentiable at $t=0$ with $\frac{\partial\widehat{\Phi}}{\partial t}(0)\in C^\infty(M, \Sigma M\otimes f_0^*TN)$. Therefore, $\Phi_t := \beta_t\circ\widehat{\Phi}_t$ fulfills the conditions required above and the proposition is proved. □

9. Non-minimality

Assume that a harmonic map $f_0 : M \to N$ is given with $\alpha(M, f_0) \neq 0$, thus in particular $\ker(D\!\!\!/^{f_0}) \neq 0$. In the previous section we have seen that, if f_0 is perturbation-minimal, then for any $\Phi \in \ker(D\!\!\!/^{f_0})$ the pair (f_0, Φ) is an uncoupled Dirac-harmonic map. Hence we have obtained a linear space of uncoupled Dirac-harmonic maps of real dimension $2\dim_{\mathbb{C}}\ker(D\!\!\!/^{f_0})$. Although we have no proof for the moment, it seems that the perturbation-minimal case is the generic one.
In this section we study the case where f_0 is not perturbation-minimal. Obviously, the proof of the perturbation-minimal setting cannot be easily adapted since the dimension of $\ker(D\!\!\!/_t)$ does no longer remain constant for small t. However, using the grading arguments explained in Section 6, we will obtain a space of uncoupled, nontrivial Dirac-harmonic maps of even larger dimension. Most of the spaces — but not all — are actually complex vector spaces. Nevertheless, for homogeneity of presentation we only use real dimensions in the theorem below.
To that extent we define, for any $m \equiv 0, 1, 2, 4$ (8), the integers b_m and d_m by the following formulas in which the minimum runs over all compact Riemannian spin manifolds M of dimension m, all spin structures on M, all compact Riemannian manifolds N and all smooth maps $f : M \to N$ with non-trivial α-index.

$$b_m := \min\Big\{ \dim_{\mathbb{R}}(\ker(D\!\!\!/^f)) \,|\, f \in C^\infty(M^m, N),\, \alpha(M, f) \neq 0\Big\}$$

and

$$d_m := \min\Big\{ \max_{\varepsilon\in\{\pm\}} \{ \dim_{\mathbb{R}}(\ker(D\!\!\!/^f_\varepsilon))\} \,|\, f \in C^\infty(M^m, N),\, \alpha(M, f) \neq 0 \text{ and } f \text{ non-perturbation-minimal}\Big\}.$$

Both b_m and d_m are positive integers because of the assumption $\alpha(M, f) \neq 0$ for all admissible M and f (and such exist in each dimension under consideration).
From now on assume $f_0 \in C^\infty(M, N)$ to be a harmonic map. By Proposition 8.2, the number b_m is a lower bound for the *real* dimension of the space of Φ_0's such that (f_0, Φ_0) is a Dirac-harmonic map for a given perturbation-minimal and harmonic map $f_0 \in C^\infty(M^m, N)$. In case the given map $f_0 \in C^\infty(M^m, N)$ is harmonic but no more perturbation-minimal, the number d_m is a lower bound for the *real* dimension of the space of Φ_0's such that (f_0, Φ_0) is Dirac-harmonic (Corollary 6.1). The main result of this section provides a lower bound for d_m. For the sake of completeness, we include the corresponding lower bound for b_m.

Theorem 9.1. *With the above notations, b_m and d_m are bounded below by the following integers:*

m mod 8	0	1	2	4
$b_m \geq$	2	2	4	4
$d_m \geq$	4	3	6	8

In particular, $b_m \geq a_m$ and $d_m \geq a_m$ where again $a_m = 2$ for $m \equiv 0, 1$ (8) and $a_m = 4$ for $m \equiv 2, 4$ (8).

Roughly speaking, the theorem says that in the non-perturbation-minimal case the space of Dirac-harmonic maps we obtain is larger than in the perturbation-minimal one. Note that Theorem 9.1 implies Theorem 1.2.

Proof. We handle the four cases separately.
• **Case** $m \equiv 0$ (8): If $\alpha(M, f_0) \neq 0$, then $|\alpha(M, f_0)| \geq 1$, thus

$$|\dim_{\mathbb{C}} \ker \not{D}^{f_0}_{+} + \dim_{\mathbb{C}} \ker \not{D}^{f_0}_{-}| \geq |\mathrm{ind}(\not{D}^{f_0}_{+})| \geq 1,$$

which implies $b_m \geq 2$. On the other hand we have

$$\max_{\varepsilon\in\{\pm\}} (\dim_{\mathbb{C}} \ker(\not{D}^{f_0}_{\varepsilon})) \quad = \quad \min_{\varepsilon\in\{\pm\}} (\dim_{\mathbb{C}} \ker(\not{D}^{f_0}_{\varepsilon})) + |\mathrm{ind}(\not{D}^{f_0}_{+})|$$

Thus, if f_0 is not perturbation-minimal, i. e. if $\ker(\not{D}^{f_0}_{\varepsilon})) \neq \{0\}$, then

$$d_m = 2\min\Big\{ \max_{\varepsilon\in\{\pm\}} (\dim_{\mathbb{C}} \ker(\not{D}^{f_0}_{\varepsilon}))\Big\} \geq 4.$$

• **Case** $m \equiv 4$ (8): If $\alpha(M, f_0) \neq 0$, then $|\alpha(M, f_0)| \geq 1$ as well. Recall that, in these dimensions there exists a quaternionic structure on the (twisted) spinor bundle commuting with both the $\mathbb{Z}_2$-grading and the Dirac operator $\not{D}^{f_0}$, see e. g. [16, Sec. 1.7]. This quaternionic structure turns the vector spaces $\Sigma_{\pm}M \otimes f_0^*TN$ into quaternionic spaces. Thus the discussion is analogous to above, but all real dimensions are divisible by 4 instead of 2. We obtain $b_m \geq 4$ and $d_m \geq 8$.
• **Case** $m \equiv 1$ (8): If $\alpha(M, f_0) \neq 0$, then $\alpha(M, f_0) = 1 \in \mathbb{Z}_2$, so that $\dim_{\mathbb{C}} \ker(\not{D}^{f_0}) \geq 1$ in case f_0 is perturbation-minimal. Hence $b_m \geq 2$. If f_0 is not perturbation-minimal, then for any $\varepsilon \in \{+, -\}$ we have $\dim_{\mathbb{R}} \ker(\not{D}^{f_0}_{\varepsilon}) = \dim_{\mathbb{C}} \ker(\not{D}^{f_0}) \geq 3$. This shows $d_m \geq 3$.
• **Case** $m \equiv 2$ (8): If $\alpha(M, f_0) \neq 0$, then $\alpha(M, f_0) = 1 \in \mathbb{Z}_2$. Recall that, in these dimensions there exists a quaternionic structure on the (twisted) spinor bundle anti-commuting with the $\mathbb{Z}_2$-grading $G \otimes \mathrm{id}$ and commuting with the Dirac operator $\not{D}^{f_0}$, see again e. g. [16, Sec. 1.7]. Thus $\ker(\not{D}^{f_0})$ is a quaternionic vector spaces and thus $\dim_{\mathbb{C}} \ker(\not{D}^{f_0}) \geq 2$ in the perturbation-minimal case, implying $b_m \geq 4$. As the quaternionic structure anti-commutes with $G \otimes \mathrm{id}$, we also have

$$\dim_{\mathbb{R}} \ker(\not{D}^{f_0}_{\varepsilon}) = \dim_{\mathbb{C}} \ker(\not{D}^{f_0})$$

for $\varepsilon = +$ and $\varepsilon = -$. If f_0 is not perturbation-minimal, then $\dim_{\mathbb{C}} \ker(\not{D}^{f_0}) \geq 6$. It follows $d_m \geq 6$. □

Remark 9.2. For a given Dirac-harmonic map (f_0, Φ_0) with harmonic mapping-component f_0, the pair $(f_0, \lambda\Phi_0)$ obviously remains Dirac-harmonic for every $\lambda \in \mathbb{C}$. In particular the space of Dirac-harmonic maps (f_0, Φ_0) with fixed harmonic mapping-component f_0 is a complex cone. Therefore, in dimension $m \equiv 1$ (8) (which is the only one where the space of Φ_0's making (f_0, Φ_0) Dirac-harmonic is *a priori* only real) and for non-perturbation-minimal harmonic maps f_0, we actually obtain a complex cone of real dimension at least 4 of Φ_0's such that (f_0, Φ_0) is Dirac-harmonic.

10. Examples of maps with non-trivial index

In this section, we still assume that M is a compact spin manifold but do not fix the spin structure since e. g. in Corollary 10.3 we want to choose it such that the index does not vanish. As a consequence the spin structure χ will no longer be suppressed in $\alpha(M, \chi, E)$.

10.1. **The case $m = 2$.** In this subsection we assume M to be a closed orientable connected surface. Such a surface always carries a spin structure χ. In general this spin structure is not unique; the space of all spin structures (up to isomorphism) is an affine $\mathbb{Z}_2$-space, modeled on the $\mathbb{Z}_2$-vector space $H^1(M, \mathbb{Z}_2)$. Thus $\chi + \alpha$ is a spin structure on M for every $\alpha \in H^1(M, \mathbb{Z}_2)$. We first compute the α-genus in terms of simpler bordism invariants of surfaces which we use to produce new examples of Dirac-harmonic maps.

Proposition 10.1. *Let $E \longrightarrow M$ be a real vector bundle of rank k over a closed connected oriented surface M with spin structure χ. Let $w_1(E) \in H^1(M, \mathbb{Z}_2)$ and $w_2(E) \in H^2(M, \mathbb{Z}_2)$ be the first and second Stiefel-Whitney classes of $E \longrightarrow M$ respectively. Then*

$$\alpha(M, \chi, E) = (k + 1) \cdot \alpha(M, \chi) + \alpha(M, \chi + w_1(E)) + w_2(E)[M]. \qquad (3)$$

Proof. The proof consists of two steps.
Claim 1: *The statement holds in case that $E \longrightarrow M$ is orientable.*
Proof of Claim 1. Since any orientable real line bundle is trivial, we may assume $k \geq 2$. First, we show that it suffices to handle the case where $M = \mathbb{C}\mathrm{P}^1$. Let indeed $D^2 \subset M$ be any embedded closed 2-disc in M. Then the loop ∂D^2 lies in the commutator subgroup $[\pi_1(M \setminus \overset{\circ}{D}{}^2), \pi_1(M \setminus \overset{\circ}{D}{}^2)]$ of $\pi_1(M \setminus \overset{\circ}{D}{}^2)$. More precisely, $M \setminus \overset{\circ}{D}{}^2$ deformation retracts onto the wedge sum of $2g$ circles, where g is the genus of M. Since any orientable real vector bundle over the circle is trivial, it is also trivial on the wedge sum of circles. This in turn implies that $E_{|_{M \setminus \overset{\circ}{D}{}^2}} \longrightarrow M \setminus \overset{\circ}{D}{}^2$ is trivial. It follows that M together with $E \longrightarrow M$ can be seen as the connected sum $M \sharp \mathbb{C}\mathrm{P}^1$, where the first factor M carries the *trivial* vector bundle $\underline{\mathbb{R}^k} := \mathbb{R}^k \times M \longrightarrow M$ and $\mathbb{C}\mathrm{P}^1$ carries some vector bundle $F \longrightarrow \mathbb{C}\mathrm{P}^1$ such that E is obtained by gluing $\underline{\mathbb{R}^k}$ together with F. Here one should pay attention to the fact that, when performing the connecting sum, the bundles on both factors have to be trivialized so that their trivializations coincide on ∂D^2. Now as in classical surgery theory (without bundles), any surgery between manifolds with bundles provides a spin-bordism for vector bundles. The invariance of α under spin-bordism then gives $\alpha(M, \chi, E) = \alpha(M, \chi, \underline{\mathbb{R}^k}) + \alpha(\mathbb{C}\mathrm{P}^1, F)$ (there is only one spin structure on $\mathbb{C}\mathrm{P}^1$, so we omit the notation χ in that case). Obviously one has $\alpha(M, \chi, \underline{\mathbb{R}^k}) = k\alpha(M, \chi, \underline{\mathbb{R}}) = k\alpha(M, \chi)$ - in particular (3) holds for trivial vector bundles (of any rank), since $w_2(\underline{\mathbb{R}^k}) = 0$. Moreover, the Stiefel-Whitney number $w_2(E)[M]$ is also a spin-bordism invariant, a fact which is analogous to Pontrjagin's theorem [25, Thm. 4.9 p.52] and which can be elementarily proved in just the same way. Therefore, $w_2(E)[M] = w_2(\underline{\mathbb{R}^k})[M] + w_2(F)[\mathbb{C}\mathrm{P}^1] = w_2(F)[\mathbb{C}\mathrm{P}^1]$. Hence we are reduced to showing $\alpha(\mathbb{C}\mathrm{P}^1, F) = w_2(F)[\mathbb{C}\mathrm{P}^1]$, which is exactly (3) for $M = \mathbb{C}\mathrm{P}^1$ since $\alpha(\mathbb{C}\mathrm{P}^1) = 0$ (the existence of a metric with positive scalar curvature

implying the vanishing of the kernel of the untwisted Dirac operator).
Since $\mathbb{C}\mathrm{P}^1$ is simply-connected, each vector bundle over $\mathbb{C}\mathrm{P}^1$ is orientable. If $k \geq 3$, then there are only two isomorphism-classes of k-ranked real vector bundles over $\mathbb{C}\mathrm{P}^1$, whereas those isomorphism classes stand in one-to-one correspondence with the integral powers of the tautological complex line bundle $\gamma_1 \longrightarrow \mathbb{C}\mathrm{P}^1$ if $k = 2$, see e. g. [32, Thm. 18.5]. Actually, it suffices to show (3) for the tautological complex line bundle $E = \gamma_1$. Consider indeed $l \cdot \gamma_1 := \bigoplus_{j=1}^{l} \gamma_1 \longrightarrow \mathbb{C}\mathrm{P}^1$ for any $l \in \mathbb{N}$, $l \geq 1$. Then $l \cdot \gamma_1$ is a $2l$-ranked real vector bundle over $\mathbb{C}\mathrm{P}^1$ and is non-trivial (its total Chern-class is $1 + l \cdot a \neq 1$, where $a \in H^2(\mathbb{C}\mathrm{P}^1, \mathbb{Z}) \cong \mathbb{Z}$ is the generator given by the tautological bundle). Moreover, $l \cdot \gamma_1$ represents up to isomorphism the *only* non-trivial $2l$-ranked real vector bundle over $\mathbb{C}\mathrm{P}^1$ if $l \geq 2$. Since trivially $\alpha(\mathbb{C}\mathrm{P}^1, l \cdot \gamma_1) = l \cdot \alpha(\mathbb{C}\mathrm{P}^1, \gamma_1)$ and $w_2(l \cdot \gamma_1) = l \cdot w_2(\gamma_1)$, we are reduced to the case where $l = 1$. In case $k = 2l+1$ is odd, the bundle $l \cdot \gamma_1 \oplus \underline{\mathbb{R}} \longrightarrow \mathbb{C}\mathrm{P}^1$ is again k-ranked and non-trivial, so it is up to isomorphism the only non-trivial k-ranked real vector bundle over $\mathbb{C}\mathrm{P}^1$. As noticed above, the spin Dirac operator on $\mathbb{C}\mathrm{P}^1$ has trivial kernel, so that $\alpha(\mathbb{C}\mathrm{P}^1, l \cdot \gamma_1 \oplus \underline{\mathbb{R}}) = \alpha(\mathbb{C}\mathrm{P}^1, l \cdot \gamma_1) = l \cdot \alpha(\mathbb{C}\mathrm{P}^1, \gamma_1)$ and, as is well-known, $w_2(l \cdot \gamma_1 \oplus \underline{\mathbb{R}}) = w_2(l \cdot \gamma_1) = l \cdot w_2(\gamma_1)$, so that again we are reduced to the case where $E = \gamma_1$. Note that the case $k = 2$ can be deduced from the case $k \geq 3$ by adding a trivial real line bundle: as we have seen above, $\alpha(M, \chi, E \oplus \underline{\mathbb{R}}) = \alpha(M, \chi, E) + \alpha(M, \chi)$ and $w_2(E \oplus \underline{\mathbb{R}}) = w_2(E)$.
It remains to show $\alpha(\mathbb{C}\mathrm{P}^1, \gamma_1) = 1 = w_2(\gamma_1)[\mathbb{C}\mathrm{P}^1]$. On the one hand, since γ_1 is a complex line bundle, $w_2(\gamma_1) = [c_1(\gamma_1)]_{\mathbb{Z}_2}$. From $c_1(\gamma_1)[\mathbb{C}\mathrm{P}^1] = -1$, we obtain $w_2(\gamma_1)[\mathbb{C}\mathrm{P}^1] = [c_1(\gamma_1)[\mathbb{C}\mathrm{P}^1]]_{\mathbb{Z}_2} = 1$. On the other hand, it follows from [17, Thm. 4.5] that $\dim_{\mathbb{C}}(\ker(D\!\!\!/^{\gamma_1}) = 2$ (beware that we tensorize over $\mathbb{R}$, so that $\Sigma T\mathbb{C}\mathrm{P}^1 \otimes_{\mathbb{R}} \gamma_1 = \{\Sigma T\mathbb{C}\mathrm{P}^1 \otimes_{\mathbb{C}} \gamma_1\} \oplus \{\Sigma T\mathbb{C}\mathrm{P}^1 \otimes_{\mathbb{C}} \gamma_1^{-1}\}$). Therefore $\alpha(\mathbb{C}\mathrm{P}^1, \gamma_1) = 1$, which concludes the proof of Claim 1. ✓
Claim 2: *The statement holds in general.*
Proof of Claim 2. By definition of the determinant bundle $\Lambda^k E \to M$, the bundle $E \oplus \Lambda^k E \to M$ is orientable. Alternatively, this follows from $w_1(E) = w_1(\Lambda^k E)$. Moreover, the spinor bundle associated to the spin structure $\chi + w_1(E)$ on M is $\Sigma M \otimes_{\mathbb{R}} \Lambda^k E$, in particular $\alpha(M, \chi, E \oplus \Lambda^k E) = \alpha(M, \chi, E) + \alpha(M, \chi, \Lambda^k E) = \alpha(M, \chi, E) + \alpha(M, \chi + w_1(E))$. We apply the proposition in the orientable case for $E \oplus \Lambda^k E$ instead of E and obtain

$$\alpha(M, \chi, E) + \alpha(M, \chi + w_1(E)) = (k+1)\alpha(M, \chi) + w_2(E \oplus \Lambda^k E).$$

Now we calculate $w_2(E \oplus \Lambda^k E) = w_2(E) + w_1(E) \cup w_1(\Lambda^k E) + w_2(\Lambda^k E)$. As $\Lambda^k E$ is of real rank 1 we have $w_2(\Lambda^k E) = 0$. According to Lemma 10.2 below $w_1(E) \cup w_1(E) = 0$. Thus $w_2(E \oplus \Lambda^k E) = w_2(E)$ which shows Claim 2 and concludes the proof of Proposition 10.1. □

Lemma 10.2. *Let M be an orientable surface and $\beta \in H^1(M, \mathbb{Z}_2)$. Then $\beta \cup \beta = 0$.*

Proof. As M is orientable, the homology group $H_k(M, \mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module for $k = 0, 1, 2$, and thus the universal coefficient theorem implies that tensoring the coefficients with $\mathbb{Z}_2$ yields isomorphisms $\mathrm{mod}\ 2 : H^k(M, \mathbb{Z}) \otimes \mathbb{Z}_2 \xrightarrow{\cong} H^k(M, \mathbb{Z}_2)$. These isomorphisms are compatible with the $\cup$-products on $H^*(M, \mathbb{Z})$ and on $H^*(M, \mathbb{Z}_2)$. Let $\overline{\beta} \in H^1(M, \mathbb{Z})$

such that $\overline{\beta} \bmod 2 = \beta$. As the cup-product on $H^*(M, \mathbb{Z})$ is skew-symmetric $\overline{\beta} \cup \overline{\beta} = 0$. Thus

$$\beta \cup \beta = (\overline{\beta} \bmod 2) \cup (\overline{\beta} \bmod 2) = (\overline{\beta} \cup \overline{\beta}) \bmod 2 = 0.$$

□

From now on and unless explicitly mentioned N will be orientable.

Corollary 10.3. *Let $f : M \to N$ be a smooth map from a closed orientable connected surface M of positive genus to an odd-dimensional orientable manifold N. Then M admits a spin structure χ such that the α-genus $\alpha(M, \chi, f) \neq 0$. In particular, each harmonic map in the homotopy class of f provides a 4-dimensional space of Dirac-harmonic maps.*

Proof. For any spin structure χ on M the identity (3) reads $\alpha(M, \chi, f) = \alpha(M, \chi) + w_2(f^*TN)[M]$ because of N being odd-dimensional and orientable. The term $w_2(f^*TN)[M]$ does not depend on the spin structure χ on M. Since the genus of M is at least one, there exists for any $x \in \mathbb{Z}_2$ at least one spin structure with α-genus x. Therefore, whatever $w_2(f^*TN)[M]$ is, $\alpha(M, \chi, f)$ can be made equal to 1 for at least one spin structure on M. The homotopy invariance of the α-genus (Corollary 4.4) concludes the proof. □

Example 10.4. The choice of spin structure χ for Corollary 10.3 depends on f through the value $w_2(f^*TN)[M]$. Choose for instance E to be an orientable but non-spin real vector bundle of rank 3 over $M := \mathbb{T}^2$, e. g. let $E := (\gamma_1 \sharp \underline{\mathbb{R}^2}) \oplus \underline{\mathbb{R}} \longrightarrow \mathbb{C}\mathrm{P}^1 \sharp \mathbb{T}^2 = \mathbb{T}^2$ where γ_1 is the tautological bundle over $\mathbb{C}P^1$ which is glued together with the trivial line bundle to give a bundle over the connected sum $\mathbb{C}\mathrm{P}^1 \sharp \mathbb{T}^2$. Let N be the total space of E, take $f_1 : M \to N$ to be the embedding by the zero-section and consider $f_2 : M \to N$ defined by $f_2(z_1, z_2) := f_1(z_1^2, z_2)$ for all $(z_1, z_2) \in \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$. Then $w_2(f_1^*TN)[M] = 1$ but $w_2(f_2^*TN)[M] = 0$.

Another straightforward consequence of Proposition 10.1 is the following

Corollary 10.5. *Let M be a compact orientable connected surface with a bounding spin structure χ, and assume that $f : M \to N$ is given. If N is orientable and $\langle w_2(TN), f_*[M] \rangle \neq 0$ (in particular N is non-spin), then $\alpha(M, \chi, f) \neq 0$.*

Next we apply Proposition 10.1 and Corollary 10.3 to produce new examples of Dirac-harmonic maps. Those examples are constructed out of harmonic maps from surfaces with spin structures such that the corresponding genus $\alpha(M, \chi, f)$ does not vanish.

Recall a result from the celebrated Sacks-Uhlenbeck paper [29], proven simultaneously by L. Lemaire [24]. Given a closed orientable surface M and a closed n-dimensional Riemannian manifold N, there exists in each homotopy class of maps from M to N an energy-minimizing (in particular harmonic) map as soon as $\pi_2(N) = 0$; if $\pi_2(N) \neq 0$ then there exists a system of generators for $\pi_2(N)$ as $\pi_1(N)$-module, each containing an energy-minimizing map from $\mathbb{S}^2 = \mathbb{C}\mathrm{P}^1$ to N.

Case $\pi_2(N) = 0$: We obtain $\alpha(\mathbb{S}^2, f) = 0$ for every map $f : \mathbb{S}^2 \longrightarrow N$ since $f^*TN \longrightarrow \mathbb{S}^2$ is trivial, therefore we cannot apply our main result for $M = \mathbb{S}^2$.
If the genus of M is at least 1 and n is odd, then Corollary 10.3 already shows the existence of a spin structure χ on M for which $\alpha(M, \chi, f) \neq 0$, whatever $f : M \longrightarrow N$ is. Thus we obtain in that case the existence of a 4-dimensional space of Dirac-harmonic maps of the form (f_0, Φ) where f_0 is a harmonic map in the homotopy class of f. This enhances the previous example [21, Thm. 2] (based on twistor-spinors) for either genus at least 2 or genus 1 and bounding spin structures on $\mathbb{T}^2$, since in both situations no non-trivial twistor-spinor is available. If n is even, then $\alpha(M, \chi, f) = w_2(f^*TN)[M]$. For $n = 2$ and N closed one has $w_2(f^*TN)[M] = 0$ since N is spin, hence our main result cannot be applied. For $n \geq 4$ the number $\alpha(M, \chi, f) \neq 0$ as soon as $f^*TN \longrightarrow M$ is non-trivial, since a $k(\geq 3)$-ranked real vector bundle over a surface is trivial if and only if its first and second Stiefel-Whitney classes vanish, see e. g. [25, Sec. 12] (in particular N must be non-spin). Note that, if $n \geq 3$ and N is non-spin, then there exists at least one M and one map f such that $f^*TN \longrightarrow M$ is non-trivial [23, Prop. II.1.12], thus providing a non-trivial Dirac-harmonic map.
Case $\pi_2(N) \neq 0$: We obtain $\alpha(\mathbb{S}^2, f) = w_2(f^*TN)[\mathbb{S}^2]$ for any map $f : \mathbb{S}^2 \longrightarrow N$. For $n = 2$ and N non-orientable (otherwise N is spin) the degree of any map $f : \mathbb{S}^2 \longrightarrow N$ has to be even since $\mathbb{S}^2$ is simply-connected, therefore $w_2(f^*TN)[\mathbb{S}^2] = \deg(f) \cdot w_2(TN)[N] = 0$, so that nothing can be said. For $n \geq 3$ the α-genus vanishes if and only if $f^*TN \longrightarrow \mathbb{S}^2$ is trivial. As an example for non-vanishing, consider the standard embedding $\mathbb{C}\mathrm{P}^1 \xrightarrow{f} \mathbb{C}\mathrm{P}^2$, $[z_1 : z_2] \longmapsto [z_1 : z_2 : 0]$. Choosing the Fubini-Study metrics of holomorphic sectional curvature 4 on both $\mathbb{C}\mathrm{P}^1$ and $\mathbb{C}\mathrm{P}^2$, the map f becomes isometric and totally geodesic, in particular harmonic. Moreover, the pull-back bundle $f^*T\mathbb{C}\mathrm{P}^2$ can be identified with $\gamma_1^{-2} \oplus \gamma_1^{-1}$, in particular $w_2(f^*T\mathbb{C}\mathrm{P}^2) = w_2(\gamma_1) \neq 0$, so that $\alpha(\mathbb{S}^2, f) = 1$. Therefore there exists a non-zero $\Phi \in C^\infty(\mathbb{S}^2, \Sigma\mathbb{S}^2 \otimes f^*T\mathbb{C}\mathrm{P}^2)$ such that (f, Φ) is Dirac-harmonic. Actually the space of such Φ's is at least (real) 12-dimensional and Φ may even be chosen not to come from any twistor-spinor on $\mathbb{S}^2$, see [3]. Note that, if one changes f into $\widehat{f} := \iota \circ \pi$, where $\iota : \mathbb{R}\mathrm{P}^2 \longrightarrow \mathbb{C}\mathrm{P}^2$ is a totally geodesic embedding and $\pi : \mathbb{S}^2 \longrightarrow \mathbb{R}\mathrm{P}^2$ is the canonical projection, then $\widehat{f}$ is harmonic but this time $\widehat{f}^*T\mathbb{C}\mathrm{P}^2$ becomes trivial, hence no non-trivial Dirac-harmonic map can be found using our methods.

In an analogous way, existence results for harmonic maps by e. g. Eells-Sampson [14] (see also [30]) or Y.L. Xin [35] (see also [36, Thm 5.11]) give Dirac-harmonic maps provided the corresponding α-genus does not vanish. We summarize the results in the following two theorems, where we assume all surfaces to be connected. Note that if we say that there is a 4-dimensional space of Dirac-harmonic maps, we do not exclude the possibility that there is even a space of higher dimension. We distinguish both cases where the homotopy class of the underlying map can be prescribed (Theorem 10.6) from the one where it cannot (Theorem 10.7).

Theorem 10.6. *Let M be a closed orientable surface of genus g and N be a closed $n(\geq 3)$-dimensional orientable Riemannian manifold. Let $f : M \longrightarrow N$ be an arbitrary smooth map.*

i) Assume $\pi_2(N) = 0$, $g > 0$ and n odd. Then there exist a harmonic map f_0 homotopic to f and at least one spin structure χ on M such that there is a 4-dimensional space of Dirac-harmonic maps of the form (f_0, Φ).

ii) Assume $\pi_2(N) = 0$, $g > 0$, n even, and $f^*TN \longrightarrow M$ non-trivial. Then there is a harmonic map f_0 homotopic to f such that for any spin structure on N there is a 4-dimensional space of Dirac-harmonic maps of the form (f_0, Φ).

Note in particular that the situation where the target manifold N has non-positive sectional curvature or, more generally, no focal points, is covered by Theorem 10.6.*i*) and *ii*), since then $\pi_2(N) = 0$.

In the case $\pi_2(N) \neq 0$, bubbling-off can happen, and in general one cannot prescribe a homotopy class $[f] \in [M, N]$. It only can be prescribed modulo $\pi_2(N)$. However similar statements follow in this case, summarized in Theorem 10.7.
In this theorem two modifications of the spin condition appear that shall be discussed now.
The second Stiefel-Whitney class of TN, denoted by $w_2(TN) \in H^2(N, \mathbb{Z}_2) = \mathrm{Hom}(H_2(N, \mathbb{Z}_2), \mathbb{Z}_2)$, can be composed with the map $\mathrm{mod}\,2 : H_2(N, \mathbb{Z}) \to H_2(N, \mathbb{Z}_2)$ and then with the Hurewicz map $h : \pi_2(N) \to H_2(N, \mathbb{Z})$. Then $w_2(TN) = 0$ if and only if N is spin, and it is an elementary exercise to show that $w_2(TN) \circ \mathrm{mod}\,2 \circ h = 0$ if and only if the universal covering $\widetilde{N}$ of N is spin.
If $w_2(TN) \circ \mathrm{mod}\,2$ vanishes, then $w_2(TN)$ factors over the Bockstein map $\beta : H_2(N, \mathbb{Z}_2) \to H_1(N, \mathbb{Z})$, i.e. there is a homomorphism $w : \mathrm{im}(\beta) \to \mathbb{Z}_2$ with $w_2(TN) = w \circ \beta$. Choose a complement Γ of the 2-torsion subgroup of $H_1(N, \mathbb{Z})$, and let $\widehat{\Gamma} \subset \pi_1(N)$ be its preimage under the Hurewicz map $\pi_1(N) \to H_1(N, \mathbb{Z})$. Let $\widehat{N} \to N$ be the covering associated to $\widehat{\Gamma}$. This is a finite covering and the number of sheets is the order of the 2-torsion group of $H_1(N, \mathbb{Z})$, thus a power of 2. Since $\mathrm{im}(\beta)$ is included in the 2-torsion subgroup of $H_1(N, \mathbb{Z})$, the composition $H_2(\widehat{N}, \mathbb{Z}) \to H_2(N, \mathbb{Z}) \overset{w_2(TN)}{\to} \mathbb{Z}_2$ vanishes, and it follows that $\widehat{N}$ is spin.
This provides a necessary condition for the vanishing of $w_2(TN) \circ \mathrm{mod}\,2$, but it is unclear to us whether this is sufficient as well.

Theorem 10.7. *Let N be a closed $n(\geq 3)$-dimensional orientable Riemannian manifold.*

iii) Assume that the universal covering $\widetilde{N}$ of N is not spin. Then there is a non-constant harmonic map $f_0 : \mathbb{S}^2 \to N$ such that f_0^*TN is non-trivial. In particular, there is a 4-dimensional space of Dirac-harmonic maps of the form (f_0, Φ).

iv) Assume that $\widetilde{N}$ is spin, even-dimensional and that $w_2(TN)$ defines a non-trivial map $H_2(N, \mathbb{Z}) \to \mathbb{Z}_2$. Then there is an oriented closed *surface M of genus $g \geq 1$ and a non-constant harmonic map $f_0 : M \to N$ such that for any spin structure on M there is a 4-dimensional space of Dirac-harmonic maps of the form (f_0, Φ).*

v) Assume that $\widetilde{N}$ is spin, odd-dimensional and that $w_2(TN)$ defines a non-trivial map $H_2(N, \mathbb{Z}) \to \mathbb{Z}_2$. Then there is an oriented closed surface M of genus $g \geq 1$, a non-constant harmonic map $f_0 : M \to N$, and at least 3 spin structures on M such that there is a 4-dimensional space of Dirac-harmonic maps of the form (f_0, Φ).

vi) Assume that $w_2(TN)$ defines a trivial map $H_2(N, \mathbb{Z}) \to \mathbb{Z}_2$ (in particular $\widetilde{N}$ is spin) and that n is odd. Then for every oriented closed surface M of genus $g \geq 1$ and for every smooth map $f : M \to N$ with $f_*[M] \notin h(\pi_2(N))$, there exists a non-constant harmonic map $f_0 : M \to N$ and at least one spin structure on M such that there is a 4-dimensional space of Dirac-harmonic maps of the form (f_0, Φ).

vii) Assume that $w_2(TN)$ defines a trivial map $H_2(N, \mathbb{Z}) \to \mathbb{Z}_2$, and that N is not spin. Then there is a non-orientable closed surface $\overline{M}$ on which there is a non-constant harmonic map $\bar{f}_0 : \overline{M} \to N$ with $\langle \overline{f}_0^* w_2(TN), [\overline{M}] \rangle \neq 0$.

$vii - \alpha$) Assume that $\overline{M}$ is not diffeomorphic to $\mathbb{R}\mathrm{P}^2$ and that n is odd. The orientation covering $\widehat{M} \to \overline{M}$ defines a non-constant harmonic map $\hat{f}_0 : \widehat{M} \to N$. Then there is at least one spin structure on $\widehat{M}$ for which there is a 4-dimensional space of Dirac-harmonic maps of the form $(\hat{f}_0, \Phi)$.

$vii - \beta$) Assume $\overline{M} = \mathbb{R}\mathrm{P}^2$. Then the induced map $\pi_1(\mathbb{R}\mathrm{P}^2) \to \pi_1(N)$ is injective. Thus $\pi_1(N)$ has an element of order 2.

In *vii*) we used $[\overline{M}]$ for the $\mathbb{Z}_2$-fundamental class of $\overline{M}$. Note that, in the cases where either $H_2(N, \mathbb{Z}) \overset{w_2(TN)}{\to} \mathbb{Z}_2$ vanishes and n is even or $\overline{M} = \mathbb{R}\mathrm{P}^2$ (case $vii - \beta$)) we cannot deduce the existence of non-trivial Dirac-harmonic maps.

Proof. We prove all cases separately.
iii) Choose any smooth map $f : \mathbb{S}^2 \to N$ with $\langle f^* w_2(TN), [\mathbb{S}^2] \rangle \neq 0$. Applying Sacks-Uhlenbeck's method (Theorem A.3), one obtains a harmonic map $f_0 : \coprod_{j=1}^l \mathbb{S}^2 \to N$ which is spin bordant to f, in particular $\langle f^* w_2(TN), [\mathbb{S}^2] \rangle = \langle f_0^* w_2(TN), [\coprod_{j=1}^l \mathbb{S}^2] \rangle = \sum_{j=1}^l \langle f_{0j}^* w_2(TN), [\mathbb{S}^2] \rangle$, where $f_{0j} : \mathbb{S}^2 \to N$ denotes the (harmonic) map induced by f_0 on the j^{th} connected component of $\coprod_{j=1}^l \mathbb{S}^2$. In particular there is at least one $j \in \{1, \ldots, l\}$ with $\langle f_{0j}^* w_2(TN), [\mathbb{S}^2] \rangle \neq 0$, which already implies that f_{0j} is non-constant. Therefore, the harmonic map $f_{0j} : \mathbb{S}^2 \to N$ satisfies $\alpha(\mathbb{S}^2, f_{0j}) = \langle f_{0j}^* w_2(TN), [\mathbb{S}^2] \rangle \neq 0$ and hence provides a 4-dimensional space of Dirac-harmonic maps.
iv) Since $H_2(N, \mathbb{Z}) \overset{w_2(TN)}{\to} \mathbb{Z}_2$ does not vanish, there exists a closed orientable surface M and a smooth map $f : M \to N$ such that $\langle f^* w_2(TN), [M] \rangle \neq 0$. Necessarily M is of positive genus because of $\widetilde{N}$ being spin. Theorem A.3 yields a harmonic map $f_0 : M \amalg \coprod_{j=1}^l \mathbb{S}^2 \to N$ which is bordant to f, in particular $\langle f^* w_2(TN), [M] \rangle = \langle f_0^* w_2(TN), [M \amalg \coprod_{j=1}^l \mathbb{S}^2] \rangle =$

$\langle f_{00}^* w_2(TN), [M]\rangle + \sum_{j=1}^{l} \langle f_{0j}^* w_2(TN), [\mathbb{S}^2]\rangle$, where $f_{00} : M \to N$ and $f_{0j} : \mathbb{S}^2 \to N$ denote the maps induced on each connected component. Again, since $\widetilde{N}$ is spin, each $\langle f_{0j}^* w_2(TN), [\mathbb{S}^2]\rangle$ vanish, so that $\langle f_{00}^* w_2(TN), [M]\rangle \neq 0$, in particular f_{00} is a non-trivial harmonic map. Because of n being even, we obtain $\alpha(M, \chi, f_{00}) = \langle f_{00}^* w_2(TN), [M]\rangle \neq 0$ for every spin structure χ on M, which shows the result in that case.
v) The only difference with *v*) lies in the formula $\alpha(M, \chi, f_{00}) = \alpha(M, \chi) + \langle f_{00}^* w_2(TN), [M]\rangle$. Choosing those spin structures χ with $\alpha(M, \chi) = 0$ (and there are at least 3 of those) one obtains the statement.
vi) For any closed orientable surface M of genus $g \geq 1$ and any smooth map $f : M \to N$ with $f_*[M] \notin h(\pi_2(N))$, Theorem A.3 yields a harmonic map $f_0 : M \to N$ which has to satisfy $\langle f_0^* w_2(TN), [M]\rangle = 0$ because of the vanishing of the map $H_2(N, \mathbb{Z}) \to \mathbb{Z}_2$. However, because only 2-spheres have possibly popped off M, the map f_0 also satisfies $f_0 * [M] \notin h(\pi_2(N))$, in particular f_0 cannot be constant. Since n is odd and $g \geq 1$, there exists at least one spin structure χ on M with $\alpha(M, \chi) = 1$, in particular $\alpha(M, \chi, f_0) = \alpha(M, \chi) = 1$ and the statement follows.
vii) In case $H_2(N, \mathbb{Z}) \overset{w_2(TN)}{\to} \mathbb{Z}_2$ vanishes but N itself is non-spin, there exists a (necessarily) non-orientable closed surface $\overline{M}$ and a smooth map $\overline{f} : \overline{M} \to N$ with $\langle \overline{f}^* w_2(TN), [\overline{M}]\rangle \neq 0$. Theorem A.2 gives a harmonic map $\overline{f}_0 : \overline{M} \to N$ with $\langle \overline{f}_0^* w_2(TN), [\overline{M}]\rangle = \langle \overline{f}^* w_2(TN), [\overline{M}]\rangle \neq 0$ (recall that none of the bubbles contribute to the second Stiefel-Whitney number because of $\widetilde{N}$ being spin). In particular $\overline{f}_0$ cannot be constant. Obviously, the induced map $\hat{f}_0$ on $\widehat{M}$ remains harmonic and non-constant. Beware however that, since $\widehat{M} \to \overline{M}$ is two-fold, $\langle \hat{f}_0^* w_2(TN), [\widehat{M}]\rangle = 2 \cdot \langle \overline{f}_0^* w_2(TN), [\overline{M}]\rangle = 0$, so that $\alpha(\widehat{M}, \chi, \hat{f}_0) = n \cdot \alpha(\widehat{M}, \chi)$ for any spin structure χ on $\widehat{M}$. In case n is odd and $\widehat{M}$ is not diffeomorphic to $\mathbb{S}^2$ (that is, $\overline{M}$ is not diffeomorphic to $\mathbb{R}\mathrm{P}^2$), there exists at least one spin structure χ on $\widehat{M}$ with $\alpha(\widehat{M}, \chi) = 1$, hence $\alpha(\widehat{M}, \chi, \hat{f}_0) = 1$, which proves $vii - \alpha$). In case $\overline{M} = \mathbb{R}\mathrm{P}^2$, assume the group homomorphim $\pi(\mathbb{R}\mathrm{P}^2) \to \pi_1(N)$ induced by $\overline{f}_0$ were trivial. Then $\overline{f}_0$ could lift to a smooth map $\tilde{f}_0 : \mathbb{R}\mathrm{P}^2 \to \widetilde{N}$ through the universal cover $\widetilde{N} \to N$. Since by assumption $\widetilde{N}$ is spin one would have $0 = \langle \tilde{f}_0^* w_2(T\widetilde{N}), [\mathbb{R}\mathrm{P}^2]\rangle = \langle \overline{f}_0^* w_2(TN), [\mathbb{R}\mathrm{P}^2]\rangle$, contradiction. Therefore $\pi(\mathbb{R}\mathrm{P}^2) \to \pi_1(N)$ is injective. This shows $vii - \beta$) and concludes the proof. □

10.2. **The case $m \geq 3$.** In higher dimensions, existence results for harmonic maps still provide non-trivial examples of Dirac-harmonic maps. Consider for instance the situation where the closed target manifold N has non-positive sectional curvature. Then there exists an energy-minimizing (hence harmonic) map in every homotopy class of smooth maps from any closed manifold M into N [14, 30]. As an application, pick *any* closed connected Riemannian spin manifold N with non-positive sectional curvature and dimension $n \equiv 1$ (8). Let N' be *any* n-dimensional closed Riemannian spin manifold with non-vanishing α-genus. In dimension $n = m = 9$ the Riemannian product of a Bott manifold with an $\mathbb{S}^1$ with non-bounding spin structure gives an example of such a manifold. Define the map f :

$N \amalg N' \to N$, $f_{|_N} := \mathrm{id}_N$ and $f_{|_{N'}} := \mathrm{cst}$. Obviously, the restricted map $f_{|_{N'}}$ being constant, the vector bundle $f^*_{|_{N'}}TN$ is trivial and hence

$$\begin{aligned} \alpha(N \amalg N', f) &= \alpha(N, f_{|_N}) + \alpha(N', f_{|_{N'}}) \\ &= \alpha(N, \mathrm{id}_N) + n\alpha(N') \\ &= \alpha(N, \mathrm{id}_N) + \alpha(N') \\ &= \alpha(N, \mathrm{id}_N) + 1. \end{aligned}$$

On the other hand, since disks are contractible and N is connected, the map f induces a (smooth) map $f^\sharp : M := N\sharp N' \to N$ which is spin bordant to f. By the spin-bordism-invariance of the α-genus, $\alpha(M, f^\sharp) = \alpha(N, \mathrm{id}_N) + 1$. Therefore, either $\alpha(N, \mathrm{id}_N) = 1$ or $\alpha(M, f^\sharp) = 1$. A 2-dimensional space of Dirac-harmonic maps is provided by (N, id_N) in the first case and by $(M, f^\sharp)$ in the second case.

Appendix A. Bubbling-off for harmonic maps

In their celebrated article [29] Sacks and Uhlenbeck showed the existence of harmonic maps under certain conditions. In particular, they explained that bubbling-off effects play an important role. However, the conclusions one may draw out of the proofs of their article are stronger than the statements of their theorem. In particular, certain relations to bordisms hold, see Theorems A.1 to A.3. As these theorems are used in our article, we will explain them and give other references that finally yield complete proofs. We also need harmonic maps defined on non-orientable surfaces, and modifications of the results of [29] also hold in this non-orientable case. In order to facilitate the comparison to [29] we adapt to their notation to a large extend. One considers maps $s : M \to N$ from a closed Riemannian surface M with metric g to a closed Riemannian manifold N with metric h.

A.1. **Non-orientability of M.** As already said above, in [29] it is claimed that M should be orientable. This assumption can be easily removed. Orientability of M is needed to define the quadratic differential ϕ in [29] and to prove [29, Lemma 1.5] which states $\partial\phi = 0$ for harmonic maps. One easily verifies that the only parts in [29] using orientation are parts of the introduction, the definition of ϕ, Lemma 1.5 and Theorem 1.6.
As a consequence, in all existence results of [29] the "orientability of M"-assumption can be removed. Theorem 1.6 provides an interpretation for harmonic maps with the special property $\phi = 0$. By slightly exchanging notation, this also works for non-orientable surfaces, as we will explain now. On Riemann surfaces quadratic differentials are in natural bijection to trace-free symmetric 2-tensors, and ∂ then turns into the divergence. Thus the condition $\partial\phi = 0$ is equivalent to saying that the g-trace-free part of s^*h has vanishing divergence. This is usually written as $\mathrm{div}(s^*h)_0 = 0$. Lemma 1.5 then says: If s is harmonic, then $(s^*h)_0$ is divergence free. Theorem 1.6 then states: If s is harmonic and if $s^*h_0 = 0$, then s is a (conformal) branched immersion. Since those conditions are independent on the orientability of M, both proofs also work in the non-orientable case.

A.2. **Bubbling-off and bordisms.** We recall further notations from [29]. For $\alpha \geq 1$ they define the energy functional $E_\alpha(s) := \int_M (1 + |ds|^2)^\alpha \, dv^M$. The map $s_0 : M \to N$ is harmonic if and only if it is a stationary point of the energy functional E_1. One can try to fix a homotopy class $[M, N]$ of maps $M \to N$ and to minimize E_1 in this class. Unfortunately the direct method in the calculus of variation fails. The reason for this is that E_1 is conformally invariant in the following sense: if g_1 and g_2 are conformal metrics with $\mathrm{vol}(M, g_1) = \mathrm{vol}(M, g_2)$ then the functional E_1 defined with respect to g_1 coincides with the functional E_1 defined for g_2. The problem is avoided if one minimizes E_α for $\alpha > 1$ instead. For $\alpha > 1$ Sacks and Uhlenbeck showed that any homotopy class $[M, N]$ contains a smooth map s_α minimizing E_α. It remains to discuss whether and in which sense the functions s_α converge to a harmonic map $s : M \to N$ for $\alpha \to 1$. An interesting bubbling-off phenomenon appears in [29]. As one often passes to subsequences, we will take a sequence $\alpha_i > 1$, $i \in \mathbb{N}$, converging to 1, and substitute $s_i := s_{\alpha_i}$.
The following theorem can be deduced from the proofs in [29]:

Theorem A.1. *Assume that M is a closed surface, and that $n = \dim N \geq 3$. Further let $s_i : M \to N$ be stationary points of E_{α_i} where $\alpha_i > 1$, $i \in \mathbb{N}$, is a sequence converging to 1, and assume that $E_{\alpha_i}(s_i)$ converges. Then after passing to a subsequence of indices i there is a harmonic map $s : M \amalg \coprod_{j=1}^{\ell} \mathbb{S}^2 \to N$ for some $\ell \in \mathbb{N} \cup \{0\}$ such that s_i converges to s in a certain sense.*

The convergence of s_i to s is as follows: there are finitely many poins $x_1, \ldots, x_l$ such that s_i converges to s in $C^1(U \setminus \{x_1, \ldots, x_l\})$, see [29, Theorem 4.4]. In x_j spheres may bubble-off. We work in isothermic coordinates around x_j, i.e. $x_i \cong 0$ is the base point, and the coordinates are defined on the neighborhood $U(x_i) \cong B_{2r}(0)$. Then there are sequences $y_i \to 0$ and $\varepsilon_i \to 0$ such that $w \mapsto s_i(\varepsilon_i w + y_i)$, defined on a ball of radius $r\varepsilon_i^{-1}$, converges to a map $\tilde{s} : \mathbb{R}^2 \to N$ in the C^1-topology. This map $\tilde{s}$ is harmonic. Conformally identifying $\mathbb{R}^2$ with $\mathbb{S}^2 \setminus \{p\}$ one obtains a harmonic map $\mathbb{S}^2 \setminus \{p\} \to N$, which can be extended to a harmonic map $\mathbb{S}^2 \to N$, see [29, Theorem 4.6]. This is the restriction of the map s above to one of the spheres $\mathbb{S}^2$. The situation is even slightly more involved as spheres can bubble-off simultaneously on different growth scales in the same point x_j. In other words it might happen that on a given bubble, a finite numbers of new bubbles emerge. And this bubbles-on-bubbles-phenomenon can be iterated to bubbles on bubbles on bubbles, etc.. However, for a given manifold N and a given upper bound on the energy (which trivially exists as we consider minimizing maps, see also [29, Prop. 2.4]), a finite number of iterations suffices. This leads to the notion of bubble tree. The precise way in which the maps s_α converge to such a bubble tree was analyzed by Parker in [27]. We also recommend [28] for a more informal introduction to bubble trees.
This precise description of the bubbling-off procedure also implies the following theorem.

Theorem A.2. *With the assumptions of the previous theorem, the subsequence can be chosen such that there is a 3-dimensional manifold W with* *boundary* $\partial W = M \amalg \coprod_{j=1}^{\ell} \mathbb{S}^2 \amalg M$ *with smooth maps* $F_i, F : W \to N$ *extending* s_i *and* s.

We will now describe the manifold W. We assume for simplicity that M is connected, otherwise one should consider each connected component separately. The manifold W is obtained by attaching ℓ copies of a 1-handle to $W_0 := (M \amalg \coprod_{j=1}^{\ell} \mathbb{S}^2) \times [0,1]$. Such an attachment is carried out inductively as follows, see e.g. [22, VI.6] for details. We combine two maps $\overline{B_1(0)} \to M$ and $\overline{B_1(0)} \to S^2$ which are diffeomorphisms onto balls in M and $\mathbb{S}^2$, to a map $G : \overline{B_1(0)} \amalg \overline{B_1(0)} \to (M \amalg \mathbb{S}^2) \times \{1\} \subset \partial W_0$. We define $W_1 := W_0 \cup_G ([-1,1] \times \overline{B_1(0)})$, where the subscript G indicates that we identify each point in $\{-1,1\} \times \overline{B_1(0)}) = \overline{B_1(0)} \amalg \overline{B_1(0)}$ with its image under G in ∂W_0. Strictly speaking W_1 is a smooth manifold with boundary and a corner, but this corner can be smoothed out by slightly changing the differentiable structure close to the corner. Then W_1 is a manifold with boundary, the boundary consists of two pieces: one piece is $(M \amalg \coprod_{j=1}^{\ell} \mathbb{S}^2) \times \{0\} \subset W_0 \subset W_1$. The remaining piece is diffeomorphic to $M \amalg \coprod_{j=1}^{\ell-1} \mathbb{S}^2$. Several choices are done here, but they do not alter the diffeomorphism type of W_1.
By attaching a further 1-handle we obtain a manifold W_2 with boundary $(M \amalg \coprod_{j=1}^{\ell} \mathbb{S}^2) \amalg (M \amalg \coprod_{j=1}^{\ell-2} \mathbb{S}^2)$, and finally after ℓ such attachments, we get $W = W_\ell$ with boundary $(M \amalg \coprod_{j=1}^{\ell} \mathbb{S}^2) \amalg M$.
If M is orientable, we even get more structure on W. In this case M also admits a spin structure, and this defines an orientation and spin structure on $M \times [0,1]$. Orientations can then be chosen on the spheres $\mathbb{S}^2$ such that the induced orientation on W_0 extends to W. Similarly the spin structure on M yields a unique spin structure on W. The manifold W then has the boundary

$$\partial W = -(M \amalg \coprod_{j=1}^{\ell} \mathbb{S}^2) \amalg M.$$

Here the minus sign indicates that this piece of the boundary carries the opposite orientation, and all pieces of the boundary carry the spin structures induced from W. Such a W is called a spin-bordism from $M \amalg \coprod_{j=1}^{\ell} \mathbb{S}^2$ to M.
For orientable surfaces we have thus strengthened the previous theorem:

Theorem A.3. *With the hypotheses of the previous theorems, assume furthermore that* M *carries a fixed orientation and spin structure. Then the* 3*-dimensional manifold* W *carries an orientation and spin structure, such that it is a spin bordism from* $M \amalg \coprod_{j=1}^{\ell} \mathbb{S}^2$ *to* M.

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