SOME EXAMPLES OF DIRAC-HARMONIC MAPS

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Abstract. We discuss a method to construct Dirac-harmonic maps developed by J. Jost, X. Mo and M. Zhu in [5]. The method uses harmonic spinors and twistor spinors, and mainly applies to Dirac-harmonic maps of codimension 1 with target spaces of constant sectional curvature. Before the present article, it remained unclear when the conditions of the theorems in [5] were fulfilled. We show that for isometric immersions into space forms, these conditions are fulfilled only under special assumptions. In several cases we show the existence of solutions.

Contents

1. Introduction and main results

Let (M^m, g) and (N^n, h) be Riemannian manifolds of dimension m and n. We assume that M carries a fixed spin structure. Note that in general we do not require M and N to be complete. Denote by ΣM the corresponding spinor bundle of M. Given a smooth map $f \colon M \longrightarrow N$, one can define the twisted Dirac-operator $D^f := \sum_{j=1}^m e_j \cdot \nabla_{e_j}^{\Sigma M \otimes f^*TN}$ acting on $C^\infty(M, \Sigma M \otimes f^*TN)$, where $(e_j)_{1 \leq j \leq m}$ is a local orthonormal frame on M and "$\cdot$" denotes Clifford multiplication $T^*M \otimes \Sigma M \otimes f^*TN \longrightarrow \Sigma M \otimes f^*TN$. Here $\Sigma M \otimes f^*TN$ is to be understood as the real tensor product of ΣM with f^*TN and is endowed with a natural Hermitian inner product $\langle \cdot\,, \cdot \rangle$ making the Clifford action of each tangent vector skew-Hermitian.
A pair $(f, \Phi) \in C^\infty(M, N) \times C^\infty(M, \Sigma M \otimes f^*TN)$ is called a *Dirac-harmonic map* if and only if the identities

$$\left| \begin{array}{ll} D^f \Phi & = 0 \\ \mathrm{tr}_g(\nabla df) & = \frac{1}{2} V_\Phi \end{array} \right. \tag{1}$$

hold on M, where $V_\Phi \in C^\infty(M, f^*TN)$ is the section of f^*TN defined by requiring

$$h(V_\Phi, Y) := \sum_{j=1}^m \langle e_j \cdot R^N_{Y, f_* e_j} \Phi, \Phi \rangle \text{ for all } Y \in f^*TN. \tag{2}$$

Recall that, since the Clifford multiplication of each tangent vector to M and the curvature tensor R^N of (N, h) act in a skew-Hermitian (resp. skew-symmetric) way, the sum $\sum_{j=1}^m \langle e_j \cdot R^N_{Y, f_* e_j} \Phi, \Phi \rangle$ is real. Here and in the following the notation

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$e_j \cdot R^N_{Y,f_*e_j}\Phi$ stands for $(e_j \cdot \otimes R^N_{Y,f_*e_j})\Phi$. Our convention for curvature tensors is $R^N_{X,Y} = [\nabla^N_X, \nabla^N_Y] - \nabla^N_{[X,Y]}$ for all tangent vectors X, Y.

In the recent years, there was a considerable interest for Dirac-harmonic maps in geometric analysis. The original motivation for studying Dirac-harmonic maps comes from physics: Dirac-harmonic maps are the fermionic analogue of the harmonic map equation. While harmonic maps are stationary points of the classical (bosonic) energy functional $f \mapsto \frac{1}{2}\int_M |df|^2\, dv^M$, Dirac harmonic maps are stationary points of the functional $(f, \Phi) \mapsto \frac{1}{2}\int_M (|df|^2 + \langle \Phi, D^f\Phi\rangle)\, dv^M$ which is interpreted as the fermionic counterpart of the classical energy functional. In geometric analysis Dirac-harmonic maps turn out to be an interesting area of investigation, as on the one hand side these equations are simple enough to allow regularity statements, removal of singularities, short-time existence of associated parabolic flows and much more, and on the other they are involved enough to exhibit a rich structure.

The goal of the article [5], written by J. Jost, X. Mo and M. Zhu was to find solutions (f, Φ) to the Dirac-harmonic map equations (1) in the form (f, Φ) where

$$\Phi := \sum_{j=1}^{m} e_j \cdot \psi \otimes f_*e_j + \varphi \otimes \nu, \tag{3}$$

such that $\psi, \varphi \in C^\infty(M, \Sigma M)$ are untwisted spinor fields and such that $\nu \in C^\infty(M, f^*TN)$ is a vector field standing orthogonally onto $df(TM) = f_*(TM)$ at each point. The first motivation for considering Dirac-harmonic maps in the form (3) is that it gives a simple way to produce Dirac-harmonic maps when M is a surface: if we assume $m = 2$, that the map f is harmonic, that ψ is a twistor spinor, $\nu = 0$, and $\varphi = 0$, then the pair (f, Φ) is a Dirac-harmonic map, see [5, Theorem 2] and Corollary 2.3 below. In particular, a lot of examples of Dirac-harmonic maps can be exhibited when M is conformally equivalent to an open subset of one of the model surfaces $\mathbb{S}^2$, $\mathbb{R}^2$ or $\mathbb{H}^2$ or conformally equivalent to a torus with trivial spin structure, we refer to Examples 2.4 for more details. On the other hand closed hyperbolic surfaces do not carry nontrivial twistor spinors and therefore this ansatz is not sufficient to construct Dirac-harmonic maps on such surfaces.

As in [5], we mainly focus in this article on the particular situation were f is an isometric immersion and $n = m+1$. This allows us — as usual for isometric immersions — to identify TM with a subbundle of f^*TN. The above ansatz then easily generates nontrivial Dirac-harmonic maps with harmonic mapping component f out of parallel spinors on M^m, see Proposition 2.5. However, the existence of nontrivial parallel spinors is very restrictive, making those examples actually very special. To get examples of Dirac-harmonic maps with non-harmonic mapping component, we furthermore assume N to be oriented with constant sectional curvature $c \in \mathbb{R}$. Note that in this case the orientations of M and N induce a global smooth unit normal vector field ν on $f(M)$; any sign convention for the choice of ν can be used, but should be fixed throughout the article. Denote by $W := -\nabla^N\nu$ the corresponding shape operator of the immersed hypersurface M and by $H := \frac{1}{m}\mathrm{tr}(W)$ its mean curvature.
We first characterize Dirac-harmonic maps of the form (3) in that setting (compare [5, Thm. 1]):

Theorem 1.1. *Let $f\colon M^m \longrightarrow N^{m+1}$ be an isometric immersion from a connected Riemannian spin manifold (M^m, g) into an oriented Riemannian manifold*

(N^{m+1}, h) with constant sectional curvature $c \in \mathbb{R}$. Let $\nu \in \Gamma(f^*TN)$ be a unit normal vector field of $f(M) \subset N$ with shape operator W and mean curvature H as explained above. For $\psi, \varphi \in C^\infty(M, \Sigma M)$ let $\Phi := \sum_{j=1}^m e_j \cdot \psi \otimes e_j + \varphi \otimes \nu$, where $(e_j)_{1\le j\le m}$ is any local orthonormal frame on M. We assume that Φ does not vanish everywhere.

i) If $m = 2$, then (f, Φ) is a Dirac-harmonic map if and only if $H = 0$, $D_M\varphi = 0$, $c \cdot \Re e(\langle\psi, \varphi\rangle) = 0$, and $e_1 \cdot \nabla^{\Sigma M}_{e_1}\psi - e_2 \cdot \nabla^{\Sigma M}_{e_2}\psi = \kappa_1\varphi$, where $We_1 = \kappa_1 e_1$. (The vector e_1 is a pointwise eigenvector for W associated to the principal curvature κ_1, we do not require e_1 to depend continuously on the basepoint.)

ii) If $m \ge 3$ and f is a totally umbilical immersion, then (f, Φ) is a Dirac-harmonic map if and only if $H = -c\Re e(\langle\psi, \varphi\rangle)$, $D_M\varphi = mH\psi$, $D_M\psi = -\frac{mH}{m-2}\varphi$ and $P\psi = 0$. If furthermore M is closed, then (f, Φ) is a Dirac-harmonic map if and only if $W = 0$, $D_M\varphi = 0$, $\nabla^{\Sigma M}\psi = 0$, and $c \cdot \Re e(\langle\psi, \varphi\rangle) = 0$.

Remark 1.2. Note that the assumption that f is totally umbilical in *ii)* already implies that H is constant, or more generally: any $m(\ge 2)$-dimensional totally umbilical hypersurface in an Einstein manifold has constant mean curvature. This is an elementary consequence of $\delta W = -mdH + \mathrm{Ric}^N(\nu)^T$, which itself follows from the Codazzi-Mainardi-identity (the 1-form $\mathrm{Ric}^N(\nu)^T \in T^*M$ is defined by $\mathrm{Ric}^N(\nu)^T(X) = h(\mathrm{Ric}^N(\nu), X)$ for all $X \in TM$). In particular, in the case $m \ge 3$ the existence of a Dirac-harmonic map (f, Φ) with $\Phi \ne 0$ given by ψ and φ as in the above theorem implies $D_M^2\psi = -\frac{m^2H^2}{m-2}\psi$ and $D_M^2\varphi = -\frac{m^2H^2}{m-2}\varphi$.

Remark 1.3. If we allowed for $\varphi = \psi = 0$, then this theorem would reduce to the classical fact that an isometric immersion is harmonic, if and only if the image has vanishing mean curvature.

Remark 1.4. Our above theorem also shows that the conditions in [5, Thm. 1] are very restrictive in the case $m \ge 3$: the authors assume the spinor field φ to be harmonic, i.e. $D_M\varphi = 0$. In this case, Theorem 1.1 yields $H = 0$ and $c\Re e(\langle\psi, \varphi\rangle) = 0$. Furthermore, $D_M\psi = 0$ and $P\psi = 0$ and this implies $\nabla^{\Sigma M}\psi = 0$. As M is isometrically immersed into N with $W = H \cdot \mathrm{Id} = 0$, it is a totally geodesic immersion, and $0 = mH \cdot \nu = \mathrm{tr}_g(\nabla df)$. In particular f is harmonic, so no example with non-harmonic map f can be produced. Assuming $\psi \not\equiv 0$, these conditions imply that M is Ricci-flat, of special holonomy and $\nabla^{\Sigma M}\varphi = 0$ as soon as M is closed.

Theorem 1.1 allows for producing new explicit examples of Dirac-harmonic maps. Denote by $N^{m+1}(c)$ any Riemannian space form of constant sectional curvature c and by $\widetilde{N}^{m+1}(c)$ the simply-connected complete Riemannian space form of constant sectional curvature c. Replacing the the metric h by $\lambda^2 h$ with a constant $\lambda > 0$ does not change the Levi-Civita connection on the tangent bundle of the target, and thus D^f is unchanged as well. In the case $c \ne 0$ we can achieve by such a rescaling with $\lambda := \sqrt{|c|}$ that the rescaled metric has sectional curvature ± 1. Thus we can assume without loss of generality $c \in \{-1, 0, 1\}$, i. e. $\widetilde{N}^{m+1}(c) = \mathbb{H}^{m+1}(-1)$, $\mathbb{R}^{m+1}$ and $\mathbb{S}^{m+1}(1)$ for $c = -1$, 0 and 1 respectively.

Theorem 1.5. *Let $f\colon M^m \to \widetilde{N}^{m+1}(c)$ be a non-minimal totally umbilical isometric immersion from a connected $m \ge 3$-dimensional Riemannian spin manifold into $\widetilde{N}^{m+1}(c)$ for some $c \in \{-1, 0, 1\}$. Then there exists a not identically vanishing Φ in the form* (3) *such that (f, Φ) is a Dirac-harmonic map if and only if $f(M)$ is an open subset of an umbilic hyperplane $\mathbb{H}^m(-\frac{4}{m+2})$ in $\widetilde{N}^{m+1}(-1) = \mathbb{H}^{m+1}(-1)$.*

The following is then an immediate consequence of the theorem by applying the theorem to the lift $\tilde{f} : \widetilde{M} \to \mathbb{H}^{m+1}(-1)$.

Corollary 1.6. *Let $f \colon M^m \to N^{m+1}$ be a non-minimal totally umbilical isometric immersion from a connected $m \geq 3$-dimensional Riemannian spin manifold into a complete connected manifold N of constant sectional curvature c. We assume that (f, Φ) is a Dirac-harmonic map, where Φ is in the form* (3)*. By Theorem 1.5 we know that $c < 0$ and by rescaling the metric on N we can achieve $c = -1$. Let $\tilde{f} : \widetilde{M} \to \widetilde{N} = \widetilde{N}^{m+1}(-1) = \mathbb{H}^{m+1}(-1)$ be a lift of f to the universal covers. Then up to isometry $\tilde{f}(\widetilde{M})$ is an open subset of a hyperplane $\mathbb{H}^m(-\frac{4}{m+2})$ in $\mathbb{H}^{m+1}(-1)$.*

Note that any connected totally umbilical isometrically immersed hypersurface in $\mathbb{H}^{m+1}(-1)$ is an open subset of some $M^m(\kappa)$ with $\kappa \geq -1$. Here $M^m(\kappa)$ is the canonically embedded complete hypersurface of constant curvature κ in $\mathbb{H}^{m+1}(-1)$, that is, $M^m(\kappa)$ is

- $\mathbb{H}^m(\kappa)$ for $\kappa \in [-1, 0)$,
- a horosphere $\mathbb{R}^m$ if $\kappa = 0$,
- the boundary of a geodesic ball if $\kappa > 0$.

The nontrivial statement in Corollary 1.6 is that only the first case can arise and that the value of κ is $-\frac{4}{m+2}$.

These notes started in 2011 as an informal comment to the authors of [5] in order to lay the basis for our article [1]. The original title was "*Examples of Dirac-harmonic maps after Jost-Mo-Zhu*". As these informal notes were cited by several authors, we decided in 2018 to transform them into a proper publication made accessible to everyone.

2. Proof of main results

The proof starts with two calculations of central importance. Denote by

$$D_M := \sum_{j=1}^{m} e_j \cdot \nabla^{\Sigma M}_{e_j} \colon C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M)$$

the classical Dirac operator by Atiyah and Singer and by

$$P \colon C^\infty(M, \Sigma M) \to C^\infty(M, T^*M \otimes \Sigma M), \quad \psi \mapsto \nabla^{\Sigma M}\psi + \frac{1}{m} \cdot \sum_{j=1}^{m} e_j^\flat \otimes e_j \cdot D_M \psi$$

the Penrose (or twistor) operator on M.

Lemma 2.1. *With the above notations, one has for f and Φ given by* (3)

$$\begin{aligned} D^f\Phi \quad = \quad & \sum_{j=1}^{m} \left(\frac{2-m}{m} e_j \cdot D_M \psi - 2P_{e_j}\psi \right) \otimes f_* e_j - \psi \otimes \mathrm{tr}_g(\nabla df) \\ & + (D_M \varphi) \otimes \nu + \sum_{j=1}^{m} e_j \cdot \varphi \otimes \nabla^N_{e_j} \nu. \end{aligned}$$

Proof. We set $\Psi := \sum_{j=1}^m e_j \cdot \psi \otimes f_* e_j$ and compute

$$\begin{aligned} D^f\Psi =& \sum_{j=1}^{m} e_j \cdot \nabla^{\Sigma M \otimes f^*TN}_{e_j} \Big(\sum_{k=1}^{m} e_k \cdot \psi \otimes f_* e_k \Big) \\ =& \sum_{j,k=1}^{m} \Big(e_j \cdot \nabla^M_{e_j} e_k \cdot \psi \otimes f_* e_k + e_j \cdot e_k \cdot \nabla^{\Sigma M}_{e_j} \psi \otimes f_* e_k \end{aligned}$$

$$+ e_j \cdot e_k \cdot \psi \otimes \nabla^{f^*TN}_{e_j} f_* e_k \Big)$$

$$\begin{aligned}
&= - \sum_{j,k=1}^m e_k \cdot e_j \cdot \nabla^{\Sigma M}_{e_j}\psi \otimes f_* e_k - 2 \sum_{j,k=1}^m \underbrace{g(e_j,e_k)}_{=\delta_{jk}} \nabla^{\Sigma M}_{e_j}\psi \otimes f_* e_k \\
&\quad + \sum_{j,k=1}^m e_j \cdot e_k \cdot \psi \otimes (\nabla df)(e_j,e_k) \\
&\quad + \underbrace{\sum_{j,k=1}^m \Big(e_j \cdot \nabla^M_{e_j} e_k \cdot \psi \otimes f_* e_k + e_j \cdot e_k \cdot \psi \otimes f_*(\nabla^M_{e_j} e_k) \Big)}_{0}.
\end{aligned}$$

Here we used $\sum_{k=1}^m \Big(e_j \cdot \nabla^M_{e_j} e_k \cdot \psi \otimes f_* e_k + e_j \cdot e_k \cdot \psi \otimes f_*(\nabla^M_{e_j} e_k) \Big) = 0$. This can either be seen by taking a frame with $\nabla^M_{e_j} e_k|_p = 0$ for some $p \in M$ and all j,k, and to calculate in p, or alternatively by writing $\nabla^M_{e_j} e_k = \sum_{\ell=1}^m \Gamma^\ell_{jk} e_\ell$ and using $\Gamma^\ell_{jk} = -\Gamma^k_{j\ell}$. Furthermore for $j \neq k$ the expression $e_j \cdot e_k \cdot \psi$ is antisymmetric for permuting j and k while $(\nabla df)(e_j,e_k)$ is symmetric, thus all terms $e_j \cdot e_k \cdot \psi \otimes (\nabla df)(e_j,e_k)$ cancel for $j \neq k$. We continue the computation:

$$\begin{aligned}
D^f \Psi &= - \sum_{k=1}^m e_k \cdot D_M \psi \otimes f_* e_k - 2 \sum_{k=1}^m \nabla^{\Sigma M}_{e_k} \psi \otimes f_* e_k - \underbrace{\sum_{k=1}^m \psi \otimes (\nabla df)(e_k,e_k)}_{=\psi \otimes \mathrm{tr}_g(\nabla df)} \\
&= \frac{2-m}{m} \sum_{k=1}^m e_k \cdot D_M \psi \otimes f_* e_k - 2 \sum_{k=1}^m P_{e_k} \psi \otimes f_* e_k - \psi \otimes \mathrm{tr}_g(\nabla df),
\end{aligned}$$

where we used the definition of P. On the other hand,

$$\begin{aligned}
D^f(\varphi \otimes \nu) &= \sum_{j=1}^m e_j \cdot \nabla^{\Sigma M \otimes f^*TN}_{e_j} (\varphi \otimes \nu) \\
&= \sum_{j=1}^m e_j \cdot \Big(\nabla^{\Sigma M}_{e_j} \varphi \otimes \nu + \varphi \otimes \nabla^{f^*TN}_{e_j} \nu \Big) \\
&= (D_M \varphi) \otimes \nu + \sum_{j=1}^m e_j \cdot \varphi \otimes \nabla^N_{e_j} \nu.
\end{aligned}$$

Using $D^f \Phi = D^f \Psi + D^f(\varphi \otimes \nu)$ this yields the claimed formula for $D^f \Phi$. □

Lemma 2.2. *Again we use the above notations, we assume that Φ is given by* (3) *and that V_Φ is given by* (2)*. Then we have for all $Y \in f^*TN$,*

$$h(V_\Phi, Y) = 2 \sum_{j,k=1}^m h(R^N_{Y, f_* e_j} f_* e_k, \nu)\, \Re e(\langle e_j \cdot e_k \cdot \psi, \varphi \rangle).$$

Proof. As in the proof of Lemma 2.2, let $\Psi := \sum_{j=1}^m e_j \cdot \psi \otimes f_* e_j$ and V_Ψ be the associated vector field as in (2). Recall that $\Phi \mapsto \sum_{j=1}^m e_j \cdot R^N_{Y, f_* e_j} \Phi$ is Hermitian, in particular

$$h(V_\Phi, Y) = h(V_\Psi, Y) + h(V_{\varphi \otimes \nu}, Y) + 2\Re e \left(\sum_{j=1}^m \langle e_j \cdot R^N_{Y, f_* e_j} \Psi, \varphi \otimes \nu \rangle \right)$$

for all $Y \in f^*TN$. We compute each term separately. First,

$$\begin{aligned} h(V_\Psi, Y) &= \sum_{j,k,\ell=1}^{m} \Re e\left(\langle e_j \cdot R^N_{Y,f_*e_j}(e_k \cdot \psi \otimes f_*e_k), e_\ell \cdot \psi \otimes f_*e_\ell\rangle\right) \\ &= \sum_{j,k,\ell=1}^{m} \Re e\left(\langle (e_j \cdot e_k \cdot \psi) \otimes R^N_{Y,f_*e_j} f_*e_k, e_\ell \cdot \psi \otimes f_*e_\ell\rangle\right) \\ &= \sum_{j,k,\ell=1}^{m} h(R^N_{Y,f_*e_j} f_*e_k, f_*e_\ell)\, \Re e\left(\langle e_j \cdot e_k \cdot \psi, e_\ell \cdot \psi\rangle\right) \\ &= -\sum_{j,k,\ell=1}^{m} h(R^N_{Y,f_*e_j} f_*e_k, f_*e_\ell)\, \Re e\left(\langle e_\ell \cdot e_j \cdot e_k \cdot \psi, \psi\rangle\right). \end{aligned}$$

This sum contains different kinds of terms. For $j = k$ we calculate

$$\Re e\left(\langle e_\ell \cdot e_j \cdot e_j \cdot \psi, \psi\rangle\right) = -\Re e\left(\langle e_\ell \cdot \psi, \psi\rangle\right) = 0$$

and with similar arguments all terms with $k = \ell$ or $j = \ell$ vanish, including $j = k = \ell$. Given a fixed triple (j, k, ℓ) with $j \neq k \neq \ell \neq j$, consider the $\mathbb{Z}/3\mathbb{Z}$-action given by the cyclic permutation sending j on k and k on ℓ. Then the sum corresponding to the $\mathbb{Z}/3\mathbb{Z}$-orbit vanishes: by definition of the Clifford multiplication,

$$e_\ell \cdot e_j \cdot e_k = e_k \cdot e_\ell \cdot e_j = e_j \cdot e_k \cdot e_\ell,$$

so that, using the first Bianchi identity for the curvature tensor of (N, h),

$$\begin{aligned} &\sum_{\sigma \in \mathbb{Z}/3\mathbb{Z}} h(R^N_{Y,f_*e_{\sigma(j)}} f_*e_{\sigma(k)}, f_*e_{\sigma(\ell)})\, \Re e\left(\langle e_{\sigma(\ell)} \cdot e_{\sigma(j)} \cdot e_{\sigma(k)} \cdot \psi, \psi\rangle\right) \\ &= \left(h(R^N_{Y,f_*e_j} f_*e_k, f_*e_\ell) + h(R^N_{Y,f_*e_k} f_*e_\ell, f_*e_j) + h(R^N_{Y,f_*e_\ell} f_*e_j, f_*e_k)\right) \\ &\quad \cdot \Re e\left(\langle e_\ell \cdot e_j \cdot e_k \cdot \psi, \psi\rangle\right) \\ &= 0. \end{aligned}$$

Therefore, $V_\Psi = 0$.
For $\varphi \otimes \nu$, using $h(R^N_{Y,f_*e_j}\nu, \nu) = 0$, we obtain

$$\begin{aligned} h(V_{\varphi\otimes\nu}, Y) &= \sum_{j=1}^{m} \Re e\left(\langle (e_j \cdot \varphi) \otimes R^N_{Y,f_*e_j}\nu, \varphi \otimes \nu\rangle\right) \\ &= \sum_{j=1}^{m} h(R^N_{Y,f_*e_j}\nu, \nu)\, \Re e\left(\langle e_j \cdot \varphi, \varphi\rangle\right) \\ &= 0, \end{aligned}$$

so that $V_{\varphi\otimes\nu} = 0$. As for the cross term, we obtain

$$\begin{aligned} \Re e\left(\sum_{j=1}^{m} \langle e_j \cdot R^N_{Y,f_*e_j}\Psi, \varphi \otimes \nu\rangle\right) &= \sum_{j,k=1}^{m} \Re e\left(\langle (e_j \cdot e_k \cdot \psi) \otimes R^N_{Y,f_*e_j} f_*e_k, \varphi \otimes \nu\rangle\right) \\ &= \sum_{j,k=1}^{m} h(R^N_{Y,f_*e_j} f_*e_k, \nu)\, \Re e\left(\langle e_j \cdot e_k \cdot \psi, \varphi\rangle\right). \end{aligned}$$

The result follows. ∎

As a straightforward consequence of Lemmata 2.1 and 2.2, we reprove [5, Theorem 2] by J. Jost, X. Mo and M. Zhu.

Corollary 2.3. *Let $m = 2$, assume that the spinor field ψ is a twistor spinor, that φ is the zero section and that the map f is harmonic. Then (f, Φ), defined by* (3)*, is a Dirac-harmonic map.*

Proof. Lemma 2.1 implies $D^f\Phi = 0$. Furthermore V_Φ vanishes since the Hermitian inner product $\langle e_j \cdot e_k \cdot \psi, e_l \cdot \psi\rangle$ is purely imaginary for all $j, k, l \in \{1, 2\}$. □

Examples 2.4.

(1) The two-dimensional round sphere $\mathbb{S}^2$ carries a 4-dimensional space of twistor spinors; a twistor spinor on $\mathbb{S}^2$ is the sum of a $\frac{1}{2}$- and of a $-\frac{1}{2}$-Killing spinor, see e.g. [2] or [4, App. A]. By Corollary 2.3, for any harmonic map $f\colon \mathbb{S}^2 \to N^n$, there exists a nonzero $\Phi \in C^\infty(\mathbb{S}^2, \Sigma\mathbb{S}^2 \otimes f^*TN)$ such that (f, Φ) is a Dirac-harmonic map.

(2) If M^2 is the flat plane $\mathbb{R}^2$ or any open subset of it, then it carries an infinite-dimensional space of twistor spinors; this space turns out to be isomorphic to the sum of the space of holomorphic functions with that of anti-holomorphic functions on M, see e.g. [4, Prop. A.2.3]. As a consequence of Corollary 2.3, for any harmonic map $f\colon M^2 \to N^n$, there exists a nonzero $\Phi \in C^\infty(M, \Sigma M \otimes f^*TN)$ such that (f, Φ) is a Dirac-harmonic map.

(3) Since the kernel of the Penrose operator as well as harmonicity of f is conformally invariant, the former for every m, see [2], and the latter only for $m = 2$, see [3], the examples described above are still valid when the metric g is chosen in the conformal class of the standard metric on M. In particular, the same kind of examples can be built on the hyperbolic plane $\mathbb{H}^2$ since it is conformally equivalent to a flat disk.

(4) Nontrivial quotients of model surfaces may carry twistor spinors, this depends on the group that is divided out but also on the spin structure chosen on the quotient. For instance, the only compact quotients of model surfaces carrying nontrivial twistor spinors are $M = \mathbb{S}^2$ and $M = \mathbb{T}^2$ where the latter carries the trivial spin structure, that is, the spin structure that is a trivial 2-fold covering of the unit circle bundle over M. This spin structure can also be characterized as the only spin structure on $\mathbb{T}^2$ which is not obtained by restricting a spin structure on a solid torus to its boundary torus $\mathbb{T}^2$. Note in particular that no nontrivial twistor spinor exists on closed hyperbolic surfaces, thus no example of the form above can be produced in that case.

It is interesting to notice another consequence of Lemmata 2.1 and 2.2:

Proposition 2.5. *With the above notations, assume that M^m carries a nontrivial parallel spinor. Then for any Riemannian manifold N^{m+1} and any harmonic map $f\colon M^m \to N^{m+1}$, there exists a non-identically-vanishing $\Phi \in C^\infty(M, \Sigma M \otimes f^*TN)$ such that (f, Φ) is a Dirac-harmonic map.*

Proof. We let ψ be a nonzero parallel spinor on M, $\varphi = 0$ and define Φ as in (3). Because of $\varphi = 0$, $D_M\psi = 0$ and $P\psi = 0$ (any parallel spinor is both a harmonic spinor and a twistor spinor) as well as $\mathrm{tr}_g(\nabla df) = 0$ since f is harmonic, we have $D^f\Phi = 0$. On the other hand, because of $\varphi = 0$, we have $V_\Phi = 0$ by Lemma 2.2, so that $\frac{V_\Phi}{2} = 0 = \mathrm{tr}_g(\nabla df)$. Thus (f, Φ) is a Dirac-harmonic map. □

We now reformulate Lemmata 2.1 and 2.2 when $f\colon M^m \to N^n$ is an isometric immersion, $n = m + 1$, the manifold N^n is oriented. Again we identify in this case TM with a subbundle of f^*TN. Further let ν be the unit normal vector field induced by the orientations of M and N.

Proposition 2.6. *Assume f is an isometric immersion from M^m into an oriented Riemannian manifold N^{m+1}, that Φ and V_Φ are given by (3) and (2) respectively, where ν is the unit normal vector field induced by the orientations of M and N. Then one has*

$$\begin{aligned} D^f\Phi &= \sum_{j=1}^m \left(\frac{2-m}{m} e_j\cdot D_M\psi - 2P_{e_j}\psi - We_j\cdot\varphi\right)\otimes e_j \\ &\quad + (D_M\varphi - mH\psi)\otimes\nu. \end{aligned}$$

Moreover, if N has constant sectional curvature $c\in\mathbb{R}$, then $V_\Phi = -2mc\Re e\left(\langle\psi,\varphi\rangle\right)\nu$.

Proof. The identification mentioned above yields $f_*e_j = e_j$. Using $\nabla df = W\otimes\nu$, one has $\mathrm{tr}_g(\nabla df) = \mathrm{tr}(W)\nu = mH\nu$. Moreover, since $\nabla^N_X\nu = -WX$ and W is symmetric, Lemma 2.1 gives

$$\begin{aligned} D^f\Phi &= \sum_{j=1}^m \left(\frac{2-m}{m} e_j\cdot D_M\psi - 2P_{e_j}\psi\right)\otimes e_j - mH\psi\otimes\nu \\ &\quad + (D_M\varphi)\otimes\nu - \sum_{j=1}^m e_j\cdot\varphi\otimes We_j \\ &= \sum_{j=1}^m \left(\frac{2-m}{m} e_j\cdot D_M\psi - 2P_{e_j}\psi\right)\otimes e_j + (D_M\varphi - mH\psi)\otimes\nu \\ &\quad - \sum_{j,k=1}^m g(We_j,e_k)\, e_j\cdot\varphi\otimes e_k \\ &= \sum_{j=1}^m \left(\frac{2-m}{m} e_j\cdot D_M\psi - 2P_{e_j}\psi\right)\otimes e_j + (D_M\varphi - mH\psi)\otimes\nu \\ &\quad - \sum_{k=1}^m W(e_k)\cdot\varphi\otimes e_k \\ &= \sum_{j=1}^m \left(\frac{2-m}{m} e_j\cdot D_M\psi - 2P_{e_j}\psi - We_j\cdot\varphi\right)\otimes e_j \\ &\quad + (D_M\varphi - mH\psi)\otimes\nu, \end{aligned}$$

which proves the first identity. Assume now that (N^{m+1},h) has constant sectional curvature c. Then the curvature tensor of N is given for all $X,Y,Z,T\in TN$ by $h(R^N_{X,Y}Z,T) = c\cdot(h(X,T)h(Y,Z) - h(X,Z)h(Y,T))$. Lemma 2.2 yields for all $Y\in f^*TN$:

$$\begin{aligned} h(V_\Phi,Y) &= 2\sum_{j,k=1}^m h(R^N_{Y,e_j}e_k,\nu)\,\Re e(\langle e_j\cdot e_k\cdot\psi,\varphi\rangle) \\ &= 2c\cdot\sum_{j,k=1}^m \left(h(Y,\nu)\underbrace{h(e_j,e_k)}_{\delta_{jk}} - h(Y,e_k)\underbrace{h(e_j,\nu)}_{0}\right)\Re e(\langle e_j\cdot e_k\cdot\psi,\varphi\rangle) \\ &= -2mc\,h(\nu,Y)\,\Re e(\langle\psi,\varphi\rangle). \end{aligned}$$

Thus

$$h(V_\Phi + 2mc\,\Re e(\langle\psi,\varphi\rangle)\,\nu, Y) = 0 \qquad \forall\, Y\in f^*TN$$

which concludes the proof. $\square$

Now we prove Theorem 1.1. As f is isometric,∇df is the vector-valued second fundamental form of $f(M)$ in N, and we have $\mathrm{tr}_g(\nabla df) = mH \cdot \nu$. Proposition 2.6 implies that (f, Φ) is a Dirac-harmonic map if and only if $D_M\varphi = mH\psi$,

$$\frac{2-m}{m} e_j \cdot D_M\psi - 2P_{e_j}\psi - We_j \cdot \varphi = 0 \text{ for all } 1 \le j \le m \tag{4}$$

and $mH \cdot \nu = \frac{V_\Phi}{2} = -mc\Re e\left(\langle\psi, \varphi\rangle\right)\nu$. Note that taking the Clifford product of e_j with (4) and summing over j gives, using the symmetry of W,

$$\begin{aligned} 0 &= \frac{2-m}{m}\sum_{j=1}^m e_j \cdot e_j \cdot D_M\psi - 2\underbrace{\sum_{j=1}^m e_j \cdot P_{e_j}\psi}_{0} - \sum_{j=1}^m e_j \cdot We_j \cdot \varphi \\ &= (m-2)D_M\psi + mH\varphi. \end{aligned} \tag{5}$$

Case $m = 2$: Then it follows from (5) that $H\varphi = 0$. Since on the open set $\Omega := \{x \in M \,|\, H(x) \neq 0\}$ the spinor φ has to vanish, so does ψ on Ω because of $D_M\varphi = mH\psi$, so that $\Phi = 0$ on Ω and therefore on M by the unique continuation property for elliptic self-adjoint differential operators. Since we look for a pair (f, Φ) with $\Phi \neq 0$, we necessarily have $\Omega = \varnothing$, that is, $H = 0$ on M. The identities $D_M\varphi = mH\psi$, $H = -c\Re e\left(\langle\psi, \varphi\rangle\right)$ become $D_M\varphi = 0$ and $c\Re e\left(\langle\psi, \varphi\rangle\right) = 0$ respectively. Taking the Clifford product of e_j with (4) and recalling the definition of P, one obtains

$$\begin{aligned} e_j \cdot We_j \cdot \varphi &= -2e_j \cdot P_{e_j}\psi \\ &= -2e_j \cdot \nabla^{\Sigma M}_{e_j}\psi + D_M\psi \end{aligned}$$

for both $j \in \{1, 2\}$. The difference of this equation for $j = 1$ and the one for $j = 2$ yields $e_2 \cdot We_2 \cdot \varphi - e_1 \cdot We_1 \cdot \varphi = 2(e_1 \cdot \nabla^{\Sigma M}_{e_1}\psi - e_2 \cdot \nabla^{\Sigma M}_{e_2}\psi)$. Take now $(e_j)_{1\le j\le 2}$ to be a pointwise orthonormal basis of T_xM made of eigenvectors for W for some fixed $x \in M$. With the condition $H = 0$ one can write $We_1 = \kappa_1 e_1$ and $We_2 = -\kappa_1 e_2$, therefore one obtains

$$2(e_1 \cdot \nabla^{\Sigma M}_{e_1}\psi - e_2 \cdot \nabla^{\Sigma M}_{e_2}\psi) = 2\kappa_1\varphi. \tag{6}$$

As (6) implies (4) trivially, this shows *i*).

Case $m \ge 3$: It follows from (5) that $D_M\psi = -\frac{mH}{m-2}\varphi$. As a consequence, the assumption $W = H \cdot \mathrm{Id}$ (total umbilicity of f) makes (4) equivalent to $P\psi = 0$. This proves the general case. We now specialize to the case that M is closed. Then $D_M^2\psi = -\frac{m^2H^2}{m-2}\psi$ and $D_M^2\varphi = -\frac{m^2H^2}{m-2}\varphi$, see Remark 1.2. Since D_M^2 is a non-negative operator, it does not have any negative eigenvalue on a closed manifold, therefore $\psi = \varphi = 0$ unless $H = 0$, which is the only possibility because of $\Phi \neq 0$. Therefore H — hence W — has to vanish on M. Since both $D_M\psi = 0$ and $P\psi = 0$, one obtains $\nabla^{\Sigma M}\psi = 0$ (hence ψ is actually parallel). This shows *ii*) and concludes the proof of Theorem 1.1.

We now prove Theorem 1.5. Let $f\colon M^m \to \widetilde{N}^{m+1}(c)$ be a totally umbilical immersion with $m \ge 3$ and $W = H \cdot \mathrm{Id} \neq 0$. Assume the pair (f, Φ) to be Dirac-harmonic. Recall that then M has to be noncompact (Theorem 1.1). Since $P\psi = 0$, we know that $D_M^2\psi = \frac{mS_g}{4(m-1)}\psi$, where S_g is the scalar curvature of (M^m, g), see e.g. [2] or [4, Prop. A.2.1]. Comparing with $D_M^2\psi = -\frac{m^2H^2}{m-2}\psi$ and assuming $\psi \neq 0$ (otherwise $\varphi = 0$ hence $\Phi = 0$, as we have seen above), we obtain $\frac{mS_g}{4(m-1)} = -\frac{m^2H^2}{m-2}$ and the Gauß equation $S_g = m(m-1)c + m^2H^2 - |W|^2 = m(m-1)(H^2 + c)$ implies $H^2 = -\frac{m-2}{m+2}c$, in particular c must be negative, w.l.o.g. $c = -1$. Therefore

$\widetilde{N}^{m+1}(c) = \mathbb{H}^{m+1}(-1)$. In that case, $f(M)$ must be an open subset of a totally umbilical (but non-totally geodesic) hyperbolic hyperplane of constant sectional curvature $H^2+c = \frac{4}{m+2}c = -\frac{4}{m+2} < 0$. Up to changing ν into $-\nu$, one can assume H to be positive, so that $H = \sqrt{\frac{m-2}{m+2}}$. Now the space of twistor spinors on any hyperbolic space is explicitly known: it is the direct sum of the space of Killing spinors for the opposite (imaginary) Killing constants. More precisely $\ker(P) = \mathcal{K}_p \oplus \mathcal{K}_m$ on M, where $\mathcal{K}_p := \{\psi \in C^\infty(M,\Sigma M)\,|\,\nabla_X^{\Sigma M}\psi = \frac{i}{\sqrt{m+2}}X\cdot\psi\ \forall X \in TM\}$ and $\mathcal{K}_m := \{\psi \in C^\infty(M,\Sigma M)\,|\,\nabla_X^{\Sigma M}\psi = -\frac{i}{\sqrt{m+2}}X\cdot\psi\ \forall X \in TM\}$. Looking for ψ in the form $\psi = \psi_p + \psi_m$ with *a priori* arbitrary $(\psi_p,\psi_m) \in \mathcal{K}_p \oplus \mathcal{K}_m$, we write the equations of Theorem 1.1 down: one has $D_M\psi = -\frac{im}{\sqrt{m+2}}(\psi_p - \psi_m)$, in particular one has to choose $\varphi := -\frac{m-2}{mH}D_M\psi = i\sqrt{m-2}(\psi_p-\psi_m)$. The formulas for ψ and φ immediately imply $D_M\varphi = mH\psi$. The only remaining condition having to be satisfied is $H = -c\cdot\Re e(\langle\psi,\varphi\rangle)$, that is,

$$\begin{aligned}\sqrt{\frac{m-2}{m+2}} &= \sqrt{m-2}\cdot\Re e(-i\langle\psi_p+\psi_m,\psi_p-\psi_m\rangle)\\ &= \sqrt{m-2}\cdot\Im m(|\psi_p|^2-|\psi_m|^2+\langle\psi_m,\psi_p\rangle-\langle\psi_p,\psi_m\rangle)\\ &= -2\sqrt{m-2}\cdot\Im m(\langle\psi_p,\psi_m\rangle),\end{aligned}$$

that is, $\Im m(\langle\psi_p,\psi_m\rangle) = -\frac{1}{2\sqrt{m+2}}$. Note that the inner product $\langle\psi_p,\psi_m\rangle$ is anyway constant on M (its first derivative vanishes). Evaluation at a point $x \in M$ yields linear maps $\mathrm{ev}_x^p\colon \mathcal{K}_p \to \Sigma_x M$ and $\mathrm{ev}_x^m\colon \mathcal{K}_m \to \Sigma_x M$ that are both injective (an imaginary Killing spinor is a parallel section w.r.t. a modified connection) and surjective (the hyperbolic space has the maximal possible number of imaginary Killing spinors). Let $\tilde{\psi}_p := \mathrm{ev}_x^p(\psi_p)$ and $\tilde{\psi}_m := \mathrm{ev}_x^m(\psi_m)$. So in order to classify all admissible pairs (ψ_p,ψ_m) it is sufficient to classify all pairs $(\tilde{\psi}_p,\tilde{\psi}_m)$ in $\Sigma_x M$ with $\Im m(\langle\tilde{\psi}_p,\tilde{\psi}_m\rangle) = -\frac{1}{2\sqrt{m+2}}$. This is easy: for each non-zero $\tilde{\psi}_p \in \Sigma_x M$, let $\tilde{\psi}_m := \frac{i}{2\sqrt{m+2}|\tilde{\psi}_p|^2}\tilde{\psi}_p$, then $\Im m(\langle\tilde{\psi}_p,\tilde{\psi}_m\rangle) = -\frac{1}{2\sqrt{m+2}}$ and obviously all admissible pairs are of the form $(\tilde{\psi}_p, \frac{i}{2\sqrt{m+2}|\tilde{\psi}_p|^2}\tilde{\psi}_p + \chi)$ where χ runs over the real hyperplane of $\Sigma_x M$ defined by the equation $\Im m(\langle\tilde{\psi}_p,\chi\rangle) = 0$.
This concludes the proof of Theorem 1.5.

3. Concluding remarks

It may be interesting to know whether 2-dimensional examples with $\varphi \neq 0$ can be obtained from Theorem 1.1. Namely if one considers the Clifford torus $M^2 := \mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}})$ sitting canonically in $N := \mathbb{S}^3$, then the inclusion map is minimal (with principal curvatures 1 and -1), but the following short argument shows that the only Dirac-harmonic maps (f,Φ) in the form (3) have vanishing φ-component. Note at first that on the flat two-torus M the Schrödinger-Licherowicz formula implies using $D_M\varphi = 0$ that φ is parallel. Thus the statement is immediate if M carries one of the three spin structures that do not allow for nonzero parallel spinor fields on the two-torus M. But even if the spin structure on M^2 is the one admitting parallel spinors, then any Dirac-harmonic map (f,Φ) in the form (3) must have $\varphi = 0$ due to the following reason. We know from Theorem 1.1 that (f,Φ) is a Dirac-harmonic map if and only if $H = 0$ (which is the case here), $D_M\varphi = 0$, $c\cdot\Re e(\langle\psi,\varphi\rangle) = 0$, and $e_1\cdot\nabla_{e_1}^{\Sigma M}\psi - e_2\cdot\nabla_{e_2}^{\Sigma M}\psi = \kappa_1\varphi$, where $We_1 = \kappa_1 e_1$. As mentioned above $D_M\varphi = 0$ is equivalent to φ being parallel. But, taking into account that, in the particular example of the embedding $M^2 \hookrightarrow \mathbb{S}^3$, the principal curvature κ_1 is constant and the vector fields e_1, e_2 are globally defined and *parallel* on M^2, we have, differentiating w.r.t. e_1:

$$0 = e_1 \cdot \nabla^{\Sigma M}_{e_1} \nabla^{\Sigma M}_{e_1} \psi - e_2 \cdot \nabla^{\Sigma M}_{e_1} \nabla^{\Sigma M}_{e_2} \psi$$

and in the same way $0 = e_1 \cdot \nabla^{\Sigma M}_{e_2} \nabla^{\Sigma M}_{e_1} \psi - e_2 \cdot \nabla^{\Sigma M}_{e_2} \nabla^{\Sigma M}_{e_2} \psi$. By $R^{\Sigma M} = 0$ and $[e_1, e_2] = 0$, we have $\nabla^{\Sigma M}_{e_2} \nabla^{\Sigma M}_{e_1} \psi = \nabla^{\Sigma M}_{e_1} \nabla^{\Sigma M}_{e_2} \psi$, so that

$$\begin{aligned}
0 &= e_1 \cdot \nabla^{\Sigma M}_{e_1} \nabla^{\Sigma M}_{e_1} \psi - e_2 \cdot \nabla^{\Sigma M}_{e_2} \nabla^{\Sigma M}_{e_1} \psi \\
&= e_1 \cdot \nabla^{\Sigma M}_{e_1} \nabla^{\Sigma M}_{e_1} \psi - e_2 \cdot (-e_1 \cdot e_2 \cdot \nabla^{\Sigma M}_{e_2} \nabla^{\Sigma M}_{e_2} \psi) \\
&= e_1 \cdot \big(\nabla^{\Sigma M}_{e_1} \nabla^{\Sigma M}_{e_1} \psi + \nabla^{\Sigma M}_{e_2} \nabla^{\Sigma M}_{e_2} \psi \big) \\
&= e_1 \cdot (\nabla^{\Sigma M})^* \nabla^{\Sigma M} \psi \qquad \text{since } \nabla_{e_i} e_i = 0,
\end{aligned}$$

so that $(\nabla^{\Sigma M})^* \nabla^{\Sigma M} \psi = 0$, that is, $\nabla^{\Sigma M} \psi = 0$. In turn, this implies $\kappa_1 \varphi = 0$ and therefore $\varphi = 0$ because of $\kappa_1 \neq 0$. Actually we have shown that (f, Φ) in the form (3) is a Dirac-harmonic map if and only if $\varphi = 0$ and ψ is parallel.
In the case where $m = 2$ no non-trivial example of Dirac-harmonic maps from a closed hyperbolic surface can be obtained with Corollary 2.3, since those do not carry non-zero twistor spinors. In that setting, examples can be produced with the help of index-theoretical methods, see e.g. [1]. Curvature conditions implying the vanishing of the Φ defined in (3) have been investigated by X. Mo [6] and confirm that only few examples of that special form can be expected.

For higher codimensions the same approach can probably be carried out, the existence of a global unit normal ν already restricting the generality. On the other hand, there are in that case obvious examples of Dirac-harmonic maps which are *not* in the form (3): take e.g. $M := \mathbb{S}^2 = \mathbb{C}\mathrm{P}^1$ embedded totally geodesically into $N = \mathbb{C}\mathrm{P}^2$, then we know by the index theorem (see e.g. [1]) that $\dim_{\mathbb{C}}(\ker(D^f)) \equiv 2\ (4)$ and is at least 4-dimensional by [5] (the space of twistor spinors on $\mathbb{S}^2$ injects into $\ker(D^f)$), so that it is at least - actually exactly - 6-dimensional. Now if $\Phi \in \ker(D^f)$, then it is an easy remark that w.r.t. the canonical splitting $\Phi = \Phi_+ + \Phi_-$ one has $D^f \Phi_\pm = 0$ and $V_{\Phi_\pm} = 0$, in particular (f, Φ_+) and (f, Φ_-) are Dirac-harmonic maps; since $\dim_{\mathbb{C}}(\ker(D^f_\pm)) \geq 3$ and the space of pure twistor spinors is complex 2-dimensional, there are at least one non-trivial $\Phi_+ \in \ker(D^f_+)$ and one non-trivial $\Phi_- \in \ker(D^f_-)$ such that $(f, \Phi_\pm)$ are Dirac-harmonic but do not come from any twistor spinor on $\mathbb{S}^2$.

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