Positive mass theorem for the Yamabe problem on spin manifolds

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Abstract

Let (M, g) be a compact connected spin manifold of dimension $n \geq 3$ whose Yamabe invariant is positive. We assume that (M, g) is locally conformally flat or that $n \in \{3, 4, 5\}$. According to a positive mass theorem by Schoen and Yau the constant term in the asymptotic development of the Green's function of the conformal Laplacian is positive if (M, g) is not conformally equivalent to the sphere. The proof was simplified by Witten with the help of spinors. In our article we will give a proof which is even considerably shorter. Our proof is a modification of Witten's argument, but no analysis on asymptotically flat spaces is needed.

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1 Introduction

The positive mass conjecture is a famous and difficult problem which originated in physics. The mass is a Riemannian invariant of an asymptotically flat manifold of dimension $n \geq 3$ and of order $\tau > \frac{n-2}{2}$. The problem consists of proving that the mass is positive if the manifold is not conformally diffeomorphic to $(\mathbb{R}^n, \mathrm{can})$. Two good references on this subject are [LP87, Her98].

Schoen and Yau [Sch89, SY79] gave a proof if the dimension is at most 7 and Witten [Wit81, Bar86] proved the result if the manifold is spin. The positivity of the mass has been proved in several other particular cases (see e.g. [Sch84]), but the conjecture in its full generality still remains open.

This problem played an important role in geometry because its solution led to the solution of the Yamabe problem. Namely, let (M, g) be a compact connected Riemannian manifold of dimension $n \geq 3$. In [Yam60] Yamabe attempted to show that there is a metric $\tilde{g}$ conformal to g such that the scalar curvature $Scal_{\tilde{g}}$ of $\tilde{g}$ is constant. However, Trudinger realized that Yamabe's proof contained a serious gap. It was the achievement of many mathematicians to finally solve the problem of finding a conformal metric $\tilde{g}$ with constant scalar curvature. The problem of finding a conformal $\tilde{g}$ with constant scalar curvature is called the Yamabe problem. As a first step, Trudinger [Tru68] was able to repair the gap if a conformal invariant named the Yamabe invariant is non-positive. The problem is much more difficult if the Yamabe invariant is positive, which is equivalent to the existence of a metric of positive scalar curvature in the conformal class of g. Aubin [Aub76] solved the problem when $n \geq 6$ and M is not locally conformally flat. Then, in [Sch84], Schoen completed the proof that a solution to the Yamabe problem exists by using the positive mass theorem in the remaining cases. Namely, assume that (M, g) is locally conformally flat or $n \in \{3, 4, 5\}$. Let

$$L_g = \frac{4(n-1)}{n-2}\Delta_g + Scal_g$$

be the conformal Laplacian of the metric g and $P \in M$. There exists a smooth function Γ, the so-called Green's function of L_g, which is defined on $M - \{P\}$ such that $L_g\Gamma = \delta_P$ in the sense of distributions (see for example [LP87]). Moreover, if we let $r = d_g(.,P)$, then in conformal normal coordinates Γ has the following expansion at P:

$$\Gamma(x) = \frac{1}{4(n-1)\omega_{n-1}\, r^{n-2}} + A + \alpha(x) \qquad \omega_{n-1} = \mathrm{vol}(S^{n-1})$$

where $A \in \mathbb{R}$. In addition, α is a function defined on a neighborhood of P and $\alpha(0) = 0$. On this neighborhood of P, the function α is smooth if (M, g) is locally conformally flat, and it is a Lipschitz function for $n = 3, 4, 5$. Hence, in both cases $\alpha = O(r)$. Schoen has shown in [Sch84] that the positivity of A would imply the solution of the Yamabe problem. He also proved that A is a positive multiple of the mass of the asymptotically flat manifold $(M, \Gamma^{\frac{4}{n-2}} g)$. Hence, in these special cases the solution of the Yamabe problem follows from the positive mass theorem, which was proven by Schoen and Yau in [SY79, SY88].

In our article, we will give a short proof for the positivity of the constant term A in the development of the Green's function in case that M is spin and locally conformally flat. The statement of this paper is weaker than the results of Witten [Wit81] and Schoen and Yau [Sch89, SY79]. The proof in our paper is inspired by Witten's reasoning, but we have considerably simplified many of the analytic arguments. Witten's argument is based on the construction of a test spinor on the stereographic blowup which is both harmonic and asymptotically constant. We show that the Green's function for the Dirac operator on M can be used to construct such a test spinor. In this way, we obtain a very short solution of the Yamabe problem using only elementary and well known facts from analysis on compact manifolds.

The last section shows how to adapt our proof to arbitrary spin manifolds of dimensions 3, 4 and 5. In dimension 3 the proof is completely analogous. However, in dimensions 4 and 5, additional estimates have to be derived in order to get sufficient control on the Green's function of the Dirac operator.

2 The locally conformally flat case

In this section, we will assume that (M, g) is a compact, connected, locally conformally flat spin manifold of dimension $n \geq 3$. The Dirac operator is denoted by D. A spinor ψ is called *D-harmonic* if $D\psi \equiv 0$. As the solution of the Yamabe problem in the case of non-negative Yamabe invariant follows from [Tru68] we will assume that the Yamabe invariant is positive. Hence, the conformal class contains a positive scalar curvature metric. As $\dim \ker D$ is conformally invariant, we see that $\dim \ker D = 0$. We fix a point $P \in M$. We can assume that g is flat in a small ball $B_P(\delta)$ of radius δ about P, and that δ is smaller than the injectivity radius. Let $(x^1, \ldots, x^n)$ denote local coordinates on $B_P(\delta)$. On $B_P(\delta)$ we trivialize the spinor bundle via parallel transport.

Lemma 2.1. *Let $\psi_0 \in \Sigma_P M$. Then there is a D-harmonic spinor ψ on $M \setminus \{P\}$ satisfying*

$$\psi|_{B_P(\delta)} = \frac{x}{r^n} \cdot \psi_0 + \theta(x)$$

where $\theta(x)$ is a smooth spinor on $B_P(\delta)$.

It is not hard to see, that in the sense of distributions

$$D\psi = \omega_{n-1}\delta_P\psi_0,$$

where δ_P is the δ-function centered at P. Hence, by definition, $\omega_{n-1}^{-1}\psi$ is the Green's function of the Dirac operator.

Proof. Our construction of ψ follows the construction of the Green's function G of the Laplacian in [LP87, Lemma 6.4]. Namely, we take a cut-off function η with support in $B_P(\delta)$ which is equal to 1 on $B_P(\delta/2)$. We set $\Phi = \eta \frac{1}{r^{n-1}} \frac{x}{r} \cdot \psi_0$ where ψ_0 is constant. The spinor Φ is D-harmonic on $B_P(\delta/2) \setminus \{P\}$. Outside $B_P(\delta)$ we extend Φ by zero, and we obtain a smooth spinor on $M \setminus \{P\}$. As $D\Phi|_{B_P(\delta/2)} \equiv 0$, we see that $D\Phi$ extends to a smooth spinor on M. Using the selfadjointness of D together with $\ker D = \{0\}$ we know that there is a smooth spinor θ_1 such that $D\theta_1 = -D\Phi$. Obviously, $\psi = \Phi + \theta_1$ is a spinor as claimed. □

We now show that the existence of ψ implies the positivity of A.

Theorem 2.2. *Let (M, g) be a compact connected locally conformally flat manifold of dimension $n \geq 3$. Then, the mass A of (M, g) satisfies $A \geq 0$. Furthermore, equality holds if and only if (M, g) is conformally equivalent to the standard sphere (S^n, can).*

Proof. Let ψ be given by lemma 2.1. Without loss of generality, we may assume that $|\psi_0| = 1$. Let Γ be the Green's function for L_g, and $G = 4(n-1)\omega_{n-1}\Gamma$. Using the maximum principle, it is easy to see that G is positive [LP87, Lemma 6.1]. We set

$$\widetilde{g} = G^{\frac{4}{n-2}} g.$$

Using the transformation formula for Scal under conformal changes, we obtain $\mathrm{Scal}_{\widetilde{g}} = 0$. We can identify spinors on $(M \setminus \{P\}, \widetilde{g})$ with spinors $(M \setminus \{P\}, g)$ such that the fiber wise scalar product on spinors is preserved [Hit74, Hij86]. Because of the formula for the conformal change of Dirac operators, the spinor

$$\widetilde{\psi} := G^{-\frac{n-1}{n-2}} \psi$$

is a D-harmonic spinor on $(M \setminus \{P\}, \widetilde{g})$, i.e. if we write $\widetilde{D}$ for the Dirac operator in the metric $\widetilde{g}$, we have $\widetilde{D}\widetilde{\psi} = 0$. By the Schrödinger-Lichnerowicz formula we have

$$0 = \widetilde{D}^2 \widetilde{\psi} = \widetilde{\nabla}^* \widetilde{\nabla} \widetilde{\psi} + \frac{\mathrm{Scal}_{\widetilde{g}}}{4} \widetilde{\psi} = \widetilde{\nabla}^* \widetilde{\nabla} \widetilde{\psi}.$$

Integration over $M \setminus B_P(\epsilon)$, $\epsilon > 0$ and integration by parts yields

$$0 = \int_{M \setminus B_P(\epsilon)} (\widetilde{\nabla}^* \widetilde{\nabla} \widetilde{\psi}, \widetilde{\psi}) dv_{\widetilde{g}} = \int_{M \setminus B_P(\epsilon)} |\widetilde{\nabla} \widetilde{\psi}|^2 dv_{\widetilde{g}} - \int_{S_P(\epsilon)} (\widetilde{\nabla}_{\widetilde{\nu}} \widetilde{\psi}, \widetilde{\psi}) ds_{\widetilde{g}}.$$

Here $S_P(\epsilon)$ denotes the boundary $\partial B_P(\epsilon)$, $\widetilde{\nu}$ is the unit normal vector on $S_P(\epsilon)$ with respect to $\widetilde{g}$ pointing into the ball, and $ds_{\widetilde{g}}$ denotes the Riemannian volume element of $S_P(\epsilon)$. Hence, we have proved that

$$\int_{M \setminus B_P(\epsilon)} |\widetilde{\nabla} \widetilde{\psi}|^2 dv_{\widetilde{g}} = \frac{1}{2} \int_{S_P(\epsilon)} \partial_{\widetilde{\nu}} |\widetilde{\psi}|^2 ds_{\widetilde{g}} \tag{2.3}$$

If ε is sufficiently small, we have

$$\widetilde{\nu} = -G^{-\frac{2}{n-2}} \frac{\partial}{\partial r} = -\big(\varepsilon^2 + o(\varepsilon^2)\big) \frac{\partial}{\partial r} \tag{2.4}$$

$$ds_{\widetilde{g}} = G^{\frac{2(n-1)}{n-2}} ds_g = G^{\frac{2(n-1)}{n-2}} \varepsilon^{n-1} ds = \big(\varepsilon^{-(n-1)} + o(\varepsilon^{-(n-1)})\big) ds \tag{2.5}$$

where ds stands for the volume element of (S^{n-1}, can), and

$$|\widetilde{\psi}|^2 = G^{-2\frac{n-1}{n-2}}|\psi|^2 = \left(\frac{1}{r^{n-2}} + 4(n-1)\omega_{n-1}\, A + r\alpha_1(r)\right)^{-2\frac{n-1}{n-2}} \left|\frac{1}{r^{n-1}}\frac{x}{r}\cdot\psi_0 + \theta(x)\right|^2$$

where α_1 is a smooth function. This gives

$$|\widetilde{\psi}|^2 = (1 + 4(n-1)\omega_{n-1}\, Ar^{n-2} + r^{n-1}\alpha_1(r))^{-2\frac{n-1}{n-2}} \times$$
$$\left(1 + 2r^{n-1}Re(< \frac{x}{r}\cdot\psi_0, \theta(x) >) + r^{2(n-1)}|\theta(x)|^2\right)$$

Noting that $\nabla_r(\frac{x}{r}\psi_0) = 0$, we get that on $S_P(\epsilon)$ and for ε small,

$$\partial_r|\widetilde{\psi}|^2 = -8(n-1)^2\omega_{n-1}\, A\varepsilon^{n-3} + o(\varepsilon^{n-3}) \tag{2.6}$$

Plugging (2.4), (2.5) and (2.6) into (2.3), we get that for ε small

$$0 \le \int_{M\setminus B_P(\epsilon)} |\widetilde{\nabla}\widetilde{\psi}|^2 dv_{\widetilde{g}} = 4(n-1)^2\omega_{n-1}\, A \int_{S^{n-1}} ds + o(1) \tag{2.7}$$
$$= 4(n-1)^2\omega_{n-1}^2 A + o(1) \tag{2.8}$$

This implies that $A \ge 0$.

Now, we assume that $A = 0$. Then it follows from (2.7) that $\widetilde{\nabla}\widetilde{\psi} = 0$ on $M\setminus\{P\}$ and hence, $\widetilde{\psi}$ is parallel. Since the choice of ψ_0 is arbitrary, we obtain in this way a basis of parallel spinors on $(M \setminus \{P\}, \widetilde{g})$. This implies that $(M \setminus \{P\}, \widetilde{g})$ is flat and hence isometric to euclidean space. Let $I : (M\setminus\{P\}, \widetilde{g}) \to (\mathbb{R}^n, \mathrm{can})$ be an isometry. We define $f(x) = 1 + \|I(x)\|^2/4, \quad x \in M$. Then $M \setminus \{P\}, f^{-2}\widetilde{g} = f^{-2}G^{\frac{4}{n-2}}g$ is isometric to $(S^n \setminus \{N\}, \mathrm{can})$. The function $f^{-2}G^{\frac{4}{n-2}}$ is smooth on $M\setminus\{P\}$ and can be extended continuously to a positive function on M. Hence, M is conformal to (S^n, can).

3 The case of dimensions 3, 4 and 5

Now under the assumption that the dimension of M is 3, 4 or 5 we show how to adapt the proof from the last section to the case in which M is not conformally flat. For simplicity, in this article, we work with the synchronous trivialization, but the same conclusions work with the Gauduchon-Bourguignon-trivialization [AHM03]. Let us assume that (M, g) is an arbitrary connected spin manifold of dimension $n \in \{3, 4, 5\}$. We choose any $P \in M$. After possibly replacing g by a metric conformal to g, we may assume that $\mathrm{Ric}_g(P) = 0$. Let $\rho = (x_1, \cdots, x_n)$ be a system of normal coordinates defined on U and centered at P. As before, let Γ be the Green's function of the conformal Laplacian L_g, $G = 4(n-1)\omega_{n-1}\Gamma$.

We trivialize TM and ΣM by synchronous frames e_i and α_i, i.e. by frames which are parallel along radial geodesics. We assume $e_i(P) = \partial/\partial x^i(P)$. Let us introduce a convenient notation for sections in this trivialization. For $v = \sum v^i e_i(P) : U \to T_P M \cong \mathbb{R}^n$ we define $\overline{v} : U \to TM$, $\overline{v}(q) = \sum v^i(q)e_i(q)$, i.e. v is the coordinate presentation for $\overline{v}$. Similarly, for $\psi : U \to \Sigma_P M = \Sigma\mathbb{R}^n$, $\psi = \sum \psi^i\alpha_i(P)$ we write $\overline{\psi}(q) = \sum \psi^i(q)\alpha_i(q)$. In this notation, $\overline{x}$ is the outward radial vector field whose length is the radius. Similarly, we write D for the Dirac operator on flat $\mathbb{R}^n$ and $\overline{D}$ for the Dirac operator on (M, g).

Proposition 3.1. *Let (M, g) be a compact connected spin manifold of dimension $n \in \{3, 4, 5\}$. Let $P \in M$ and $\psi_0 \in \Sigma_P M$, then there is a spinor $\Psi(\psi_0)$ which is D-harmonic on $M \setminus P$, and which has in the synchronous trivialization defined above* *the following expansion at P:*

$$\Psi(\psi_0) = \frac{\bar{x}}{r^n} \cdot \psi_0 + \Theta_1(x) \;\; \textit{if } n = 3$$

$$\Psi(\psi_0) = \frac{\bar{x}}{r^n} \cdot \psi_0 + \Theta_2(x) \;\; \textit{if } n = 4$$

$$\Psi(\psi_0) = \frac{\bar{x}}{r^n} \cdot \psi_0 + \alpha(x) + \Theta_3(x) \;\; \textit{if } n = 5$$

where $\Theta_1 \in C^{0,a}(\mathbb{R}^n)$ for all $a \in]0,1[$, where, for all $\varepsilon > 0$, $r^\varepsilon \Theta_2$, $r^\varepsilon \Theta_3$, $r^{1+\varepsilon}|\nabla \Theta_1|$, $r^{1+\varepsilon}|\nabla \Theta_2|$ and $r^{1+\varepsilon}|\nabla \Theta_3|$ are continuous on $\mathbb{R}^n$ and where α is homogeneous of order -1 and even near P.

Remark. As before, the spinor $\omega_{n-1}^{-1}\Psi(\psi_0)$ is the Green's function for the Dirac operator on M. The expansion of $\Psi(\psi_0)$ could be improved but the statement of proposition 3.1 is sufficient to adapt the proof of theorem 2.2.

Proof of positive mass theorem. Using this proposition, the proof of theorem 2.2 can easily be adapted with $\psi = \Psi(\psi_0)$. As one can check, equation (2.7) is still available in dimensions 3, 4 and 5. This is easy to see in dimensions 3 and 4. In dimension 5, we note that since α is even near P and since $\frac{x}{r}$ is an odd vector field, we have

$$Re \int_{S^{n-1}} \frac{\partial}{\partial r}(< \frac{x}{r} \cdot \psi_0, \alpha(x) >)\, ds = \int_{S^{n-1}} Re(< \frac{x}{r} \cdot \psi_0, \partial_r \alpha(x) >)\, ds = 0$$

Equation (2.7) easily follows. This proves the positive mass theorem 2.2. □

We will now prove Proposition 3.1.

Definition. Let $\alpha \in \Gamma(\Sigma(\mathbb{R}^n \setminus \{0\}))$ be a smooth spinor defined on $\mathbb{R}^n \setminus \{0\}$. For $k \in \mathbb{R}$, we say that α is *homogeneous of order k* if $\alpha(sx) = s^k \alpha(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and all $s > 0$. This is equivalent to $\partial_r \alpha = k\alpha$.

Proposition 3.2. *Let α be a spinor homogeneous of order $k \in (-n, -1)$. Then there is a spinor β, homogeneous of order $k+1$, such that $D(\beta) = \alpha$.*

Proof. Let α be a homogeneous spinor of order k. Recall that $\Gamma_D := \frac{x}{\omega_{n-1} r^n} \cdot$ is the Green's function for the Dirac operator on $\mathbb{R}^n$. We define $\beta := \Gamma_D * \alpha$, i.e.

$$\beta(x) = \frac{1}{\omega_{n-1}} \lim_{\rho \to 0} \int_{\mathbb{R}^n \setminus (B_x(\rho) \cup B_0(\rho))} \frac{x-y}{|x-y|^n} \cdot \alpha(y)\, dy. \qquad x \neq 0$$

The integral converges for $|y| \to \infty$ as $k < -1$. The limit for $\rho \to 0$ exists as $k > -n$. Similarly one sees that β is smooth, and one calculates $D(\beta) = \alpha$. A simple change of variables shows that β is homogeneous of order $k+1$.

Lemma 3.3 (Regularity Lemma)**.** *Let α be a smooth spinor on $\mathbb{R}^n \setminus \{0\}$. We assume that $D(\alpha) = O(\frac{1}{r})$ and $\partial_i D(\alpha) = O(\frac{1}{r^2})$ as $r \to 0$. Then, for all $\varepsilon > 0$, $r^\varepsilon \alpha$ and $r^{1+\varepsilon}|\nabla \alpha|$ extend continuously to $\mathbb{R}^n$.*

Proof. As the statement is local, we can assume for simplicity that α vanishes outside a ball $B_0(R)$ about 0. Since $D(\alpha) \in L^q(\mathbb{R}^n)$ for all $q < n$, from regularity theory we get that $\alpha \in H_1^q(\mathbb{R}^n)$ for all $q < n$. The Sobolev embedding theorem then implies that $\alpha \in L^q(\mathbb{R}^n)$ for all $q > 1$. Moreover, we have

$$|D(r^\varepsilon \alpha)| \leq \varepsilon r^{\varepsilon - 1}|\alpha| + O(r^{\varepsilon - 1})$$

Using Hölder's inequality, we see that $D(r^\varepsilon \alpha) \in L^q(\mathbb{R}^n)$ for some $q > n$. By regularity theory, we have $r^\varepsilon \alpha \in H_1^q(\mathbb{R}^n)$ and by the Sobolev embedding theorem, $r^\varepsilon \alpha \in C^0(\mathbb{R}^n)$. This proves the first part of lemma 3.3. For the second part, we apply the same argument twice: a calculation on $\mathbb{R}^n \setminus \{0\}$ yields:

$$|D(r^{1+\varepsilon}\partial_i \alpha)| \leq (1+\varepsilon) r^\varepsilon |\partial_i \alpha| + O(r^{\varepsilon-1}) \tag{3.4}$$

In the same way, we have:

$$|D(r^{\varepsilon}\partial_i \alpha)| \leq \varepsilon r^{\varepsilon-1} |\partial_i \alpha| + O(r^{\varepsilon-2}) \tag{3.5}$$

For all $i = 1, \ldots, n$ we have $D(\partial_i \alpha) = \partial_i D\alpha = O(r^{-2})$ and $D(\partial_i \alpha) \in L^q(\mathbb{R}^n)$ for all $q < \frac{n}{2}$. Using the regularity theory and then the Sobolev embedding theorem, we get that $\partial_i \alpha \in H_1^q(\mathbb{R}^n)$ for all $q < \frac{n}{2}$ and that $\partial_i \alpha \in L^q(\mathbb{R}^n)$ for all $q < n$. The Hölder inequality implies that there is a $q > \frac{n}{2}$, close to $\frac{n}{2}$, such that $r^{\varepsilon-1}|\partial_i \alpha| \in L^q(\mathbb{R}^n)$. Together with (3.5), this shows that $D(r^\varepsilon \partial_i \alpha) \in L^q(\mathbb{R}^n)$ for some $q > \frac{n}{2}$. By the regularity and Sobolev theorems, we obtain that $r^\varepsilon \partial_i \alpha \in L^q(\mathbb{R}^n)$ for some $q > n$. Now using (3.4), we obtain $|D(r^{1+\varepsilon}\partial_i \alpha)| \in L^q(\mathbb{R}^n)$ for some $q > n$. Applying regularity theory and the Sobolev theorems again, we get that $r^{1+\varepsilon}\partial_i \alpha \in C^0(\mathbb{R}^n)$. This proves that $r^{1+\varepsilon}|\nabla \alpha| \in C^0(\mathbb{R}^n)$. □

Proof of Proposition 3.1.
For any spinor ψ, standard formulas in normal coordinates yield

$$\overline{D}(\overline{\psi}) = \overline{D\psi} + \frac{1}{4}\sum_{ijk} \Gamma_{ij}^k e_i \cdot e_j \cdot e_k \cdot \overline{\psi} \tag{3.6}$$

with $\Gamma_{ij}^k = g(\nabla_{e_i} e_j, e_k) = \sum_l \frac{1}{2} x^l g(R(e_l, e_i)e_j, e_k) + O(r^2)$. Note that Γ_{ij}^k denotes the Christoffel symbols of the second kind, which are, in general, not symmetric in i and j. The Bianchi identity implies that

$$\sum_{\substack{i,j,k \\ i \neq j \neq k \neq i}} \Gamma_{ij}^k e_i \cdot e_j \cdot e_k \in O(r^2).$$

Hence,

$$\sum_{ijk} \Gamma_{ij}^k e_i \cdot e_j \cdot e_k = \sum_l x^l \mathrm{Ric}_{lk}(P) e_k + O(r^2) = O(r^2) \tag{3.7}$$

Let η be a cut-off function equal to 1 in a neighborhood V of $P = 0$ in M, and supported in the normal neighborhood U. Let ψ_0 be a constant spinor on $\mathbb{R}^n$. We define ψ on $U \setminus \{0\}$ by

$$\overline{\psi} = \frac{\eta}{r^{n-1}} \frac{\overline{x}}{r} \cdot \overline{\psi}_0$$

and extend it with zero on $M \setminus U$. Then, ψ is smooth on $M \setminus \{P\}$ and is D-harmonic near P (see the locally conformally flat case) and by (3.6), near P we have

$$\overline{D}(\overline{\psi}) = \frac{1}{4r^{n-1}} \sum_{ijk} \Gamma_{ij}^k e_i \cdot e_j \cdot e_k \cdot \frac{\overline{x}}{r} \cdot \overline{\psi}_0.$$

Writing the Taylor development of $\sum_{ijk} \Gamma_{ij}^k e_i \cdot e_j \cdot e_k$ to order 3 at 0, we see by (3.7) that we can write $\overline{D}(\overline{\psi})$ as a sum of a spinor γ which is homogeneous of order $3-n$ and a spinor γ', smooth on $U \setminus \{0\}$, which satisfies $\gamma' = 0(r^{4-n})$ and $|\nabla \gamma'| = 0(r^{3-n})$.

$$\overline{D}(\overline{\psi}) = \gamma + \gamma'$$

If $n = 3$, $\gamma + \gamma' \in L^q(U)$ for all $q > 1$. Let η be a cut-off function as above, then $\eta(\gamma + \gamma')$ can be viewed as a spinor on $\mathbb{R}^n \setminus \{0\}$, and $\Theta := \Gamma_D * (\eta(\gamma + \gamma'))$ is a smooth spinor on $\mathbb{R}^n \setminus \{0\}$ such that $D(\Theta) = \gamma + \gamma'$ near 0. By regularity theory and the Sobolev embedding theorem, we get that $\Theta \in H_1^q(\mathbb{R}^n)$ for all $q > 1$, and hence $\Theta \in C^{0,a}(\mathbb{R}^n)$ for all $a \in (0, 1)$. Lemma 3.3 implies $r^{1+\varepsilon}|\nabla\Theta| \in C^0(\mathbb{R}^n)$.

If $n = 4$, we have $\gamma + \gamma' = 0(\frac{1}{r})$ and $|\nabla(\gamma + \gamma')| = 0(\frac{1}{r^2})$. As in dimension 3, we can find Θ, a smooth spinor on $\mathbb{R}^n \setminus \{0\}$ such that $D(\Theta) = \gamma + \gamma'$ near 0. We fix $\epsilon \in (0, 1)$. By the regularity lemma 3.3 we get $r^\varepsilon\Theta \in C^0(\mathbb{R}^n)$ and $r^{1+\varepsilon}|\nabla\Theta| \in C^0(\mathbb{R}^n)$.

If $n = 5$, by proposition 3.2, we can find a spinor α homogeneous of order -1 such that $D(\alpha) = \gamma$. Moreover, $\gamma' = 0(\frac{1}{r})$ and $|\nabla\gamma'| = 0(\frac{1}{r^2})$. Proceeding as in dimension 3 and using lemma 3.3, we can find a spinor f smooth on $\mathbb{R}^n \setminus \{0\}$ such that $D(f) = \gamma'$ near 0, such that $r^\varepsilon f \in C^0(\mathbb{R}^n)$ and such that $r^{1+\varepsilon}|\nabla f| \in C^0(\mathbb{R}^n)$. We set $\Theta := (\alpha + f)$.

Now, for all dimensions, we set

$$\varphi = \psi - \eta\Theta$$

By (3.6), we have on V

$$\overline{D}\overline{\varphi} = \overline{D}(\overline{\psi}) - \overline{D\Theta} - \frac{1}{4}\sum_{ijk} \Gamma_{ij}^k e_i \cdot e_j \cdot e_k \cdot \overline{\Theta}$$

Using (3.7) and the fact that $\overline{D}(\overline{\psi}) - \overline{D\Theta} = 0$, we get that $\overline{D}\overline{\varphi} = O(r)$ and hence is of class $C^\infty(M \setminus \{P\}) \cap C^{0,1}(M)$. As a consequence, there exists $\overline{\varphi}' \in \Gamma(\Sigma M)$ of class $C^\infty(M \setminus \{P\}) \cap C^{1,a}(M)$, $a \in (0, 1)$ such that $\overline{D}\overline{\varphi}' = \overline{D}\overline{\varphi}$. We now set $\Psi(\psi_0) = \overline{\varphi} - \overline{\varphi}'$. Proposition 3.1 follows. □

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