# MASS ENDOMORPHISM, SURGERY AND PERTURBATIONS

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Abstract. We prove that the mass endomorphism associated to the Dirac operator on a Riemannian manifold is non-zero for generic Riemannian metrics. The proof involves a study of the mass endomorphism under surgery, its behavior near metrics with harmonic spinors, and analytic perturbation arguments.

## 1. Introduction

Let $(M, g)$ be a compact Riemannian spin manifold. We always assume that a spin manifold comes equipped with a choice of orientation and spin structure. Assume that the metric $g$ is flat in a neighborhood of a point $p \in M$ and has no harmonic spinors. Then the Green's function $G^g$ at $p$ for the Dirac operator $D^g$ exists. The constant term in the expansion of $G^g$ at $p$ is an endomorphism of $\Sigma_p M$ called the mass endomorphism. The terminology is motivated by the analogy to the ADM mass which is the constant term in the Green's function of the Yamabe operator. The non-nullity of the mass endomorphism has many interesting consequences. In particular, combining the results presented here with inequalities in [7] and [14], one obtains a solution of the Yamabe problem.

Finding examples for which the mass endomorphism does not vanish is then a natural problem. In [12], see also [13], it is proven that for a generic metric on a manifold of dimension 3, the mass endomorphism does not vanish in a given point $p$. The aim of this paper is to extend this result to all dimensions at least 3, see Theorem 2.4.

## 2. Definitions and main result

The goal of this section is to give a precise statement of the main results. At first, the mass endomorphism is defined. Then, in Subsection 2.2, we define suitable sets of metrics to work with. Further, in Subsection 2.3, we explain some well known facts on the $\alpha$-genus. Finally, in Subsection 2.4 we state Theorem 2.4, which is the main result of this article.

2.1. **Mass endomorphism.** In this section we will recall the mass endomorphism introduced in [7]. Let $(M, g)$ be a compact Riemannian spin manifold of dimension $n \geq 2$ and let $p \in M$. Assume that the metric $g$ is flat in a neighborhood of $p$ and that the Dirac operator $D^g$ is invertible. The Green's function $G^g(p, \cdot) = G^g(\cdot)$ of $D^g$ at $p$ is defined by

$$D^g G^g = \delta_p \,\mathrm{Id}_{\Sigma_p M},$$

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where $\delta_p$ is the Dirac distribution at $p$ and $G^g$ is viewed as a linear map which associates to each spinor in $\Sigma_p M$ a smooth spinor field on $M \setminus \{p\}$. The distributional equation satisfied by $G^g$ should be interpreted as

$$\int_M \langle G^g(x)\psi_0, D^g\varphi(x)\rangle \, dv^g(x) = \langle \psi_0, \varphi(p)\rangle$$

for any $\psi_0 \in \Sigma_p M$ and any smooth spinor field $\varphi$. Let $\xi$ denote the flat metric on $\mathbb{R}^n$, it then holds that

$$G^\xi \psi = -\frac{1}{\omega_{n-1}|x|^n} x \cdot \psi.$$

at $p = 0$, where $\omega_{n-1}$ is defined as the volume of $S^{n-1}$. The following Proposition is proved in [7].

**Proposition 2.1.** *Let $(M, g)$ be a compact Riemannian spin manifold of dimension $n \geq 2$. Assume that $g$ is flat on a neighborhood $U$ of a point $p \in M$. Then, for $\psi_0 \in \Sigma_p M$ we have*

$$G^g(x)\psi_0 = -\frac{1}{\omega_{n-1}|x|^n} x \cdot \psi_0 + v^g(x)\psi_0,$$

*where the spinor field $v^g(x)\psi_0$ satisfies $D^g(v^g(x)\psi_0) = 0$ in a neighborhood of $p$.*

This allows us to define the mass endomorphism.

**Definition 2.2.** The *mass endomorphism* $\alpha^g : \Sigma_p M \to \Sigma_p M$ for a point $p \in U \subset M$ is defined by

$$\alpha^g(\psi_0) := v^g(p)\psi_0.$$

In particular, we have

$$\alpha^g(\psi_0) = \lim_{x \to 0} \left( G^g(x)\psi_0 + \frac{1}{\omega_{n-1}|x|^n} x \cdot \psi_0 \right).$$

The mass endomorphism is thus (up to a constant) defined as the zero order term in the asymptotic expansion of the Green's function in normal coordinates around $p$.

2.2. **Metrics flat around a point.** Let $M$ be a connected spin manifold, $p \in U$ where $U$ is an open subset of $M$. A Riemannian metric on $U$ will be called *extendible* if it possesses a smooth extension to a (not necessarily flat) Riemannian metric on $M$.

Fix a flat extendible metric $g_{\text{flat}}$ on $U$. The set of all smooth extensions of $g_{\text{flat}}$ is denoted by

$$\mathcal{R}_{U,g_{\text{flat}}}(M) := \{g \,|\, g \text{ is a metric on } M \text{ such that } g|_U = g_{\text{flat}}\}.$$

Inside this set of metrics we study those with invertible Dirac operator

$$\mathcal{R}^{\text{inv}}_{U,g_{\text{flat}}}(M) := \{g \in \mathcal{R}_{U,g_{\text{flat}}}(M) \,|\, D^g \text{ is invertible}\}.$$

The main subject of the article is the set

$$\mathcal{R}^{\neq 0}_{p,U,g_{\text{flat}}}(M) := \{g \in \mathcal{R}^{\text{inv}}_{U,g_{\text{flat}}}(M) \,|\, \text{the mass endomorphism at } p \text{ is not } 0\}.$$

Note that $\mathcal{R}^{\text{inv}}_{U,g_{\text{flat}}}(M)$ can be empty (see Subsection 2.3). We say that a subset $A \subset \mathcal{R}_{U,g_{\text{flat}}}(M)$ is *generic* in $\mathcal{R}_{U,g_{\text{flat}}}(M)$ if it is open in the $C^1$-topology and dense in the $C^\infty$-topology in $\mathcal{R}_{U,g_{\text{flat}}}(M)$.

2.3. **The $\alpha$-genus.** The $\alpha$-genus is a ring homomorphism $\alpha : \Omega_*^{\mathrm{spin}}(\mathrm{pt}) \to KO_*(\mathrm{pt})$ where $\Omega_*^{\mathrm{spin}}(\mathrm{pt})$ is the spin bordism ring and $KO_*(\mathrm{pt})$ is the ring of coefficients for $KO$-theory. In particular, the well-definedness of the map means that the $\alpha$-genus $\alpha(M)$ of a spin manifold $M$ depends only on its spin bordism class, and the homomorphism property means that it is additive with respect to the disjoint union and multiplicative with respect to the product of spin manifolds. We recall that if the dimension of $M$ is $n$ then $\alpha(M) \in KO_n(\mathrm{pt})$ and as groups we have

$$KO_n(\mathrm{pt}) \cong \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \mod 4; \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \equiv 1, 2 \mod 8; \\ 0 & \text{otherwise.} \end{cases}$$

Let $(M, g)$ be a compact Riemannian spin manifold. The Atiyah-Singer index theorem states that the Clifford index of $D^g$ coincides with $\alpha(M)$, see [16]. This implies that a manifold $M$ with $\alpha(M) \neq 0$ cannot have a metric with invertible Dirac operator. If $M$ is not connected, one can apply the argument in each connected component. Thus there are many non-connected examples $M$, with $\alpha(M) = 0$, but admitting no metric with invertible Dirac operator.

However, the converse holds true under the additional assumption that $M$ is connected, see [5]. The proof of the converse relies on a surgery construction preserving invertibility of the Dirac operator together with Stolz's examples of manifolds with positive scalar curvature in every spin bordism class [20]. Special cases were proved previously in [18] and [8]. For our purposes, it is more convenient to use a slightly stronger version, presented in [4]:

**Theorem 2.3.** *Let $M$ be a connected compact spin manifold and let $p \in M$. Let $U$ be an open subset of $M$, $p \in U \neq M$, and let $g_{\mathrm{flat}}$ be a flat extendible metric on $U$. Then $\mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(M) \neq \emptyset$ if and only if $\alpha(M) = 0$.*

Using real analyticity one obtains that $\mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(M)$ is open and dense in $\mathcal{R}_{U,g_{\mathrm{flat}}}(M)$.

2.4. **Main result.** The main result of this paper is the following: If $\alpha(M) = 0$, so that the mass endomorphism is defined for metrics in the non-empty set $\mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(M)$, then a generic metric has a non-zero mass endomorphism.

**Theorem 2.4.** *Let $M$ be a compact connected $n$-dimensional spin manifold with $n \geq 3$ and with vanishing $\alpha$-genus. Let $p \in M$ and assume that $g_{\mathrm{flat}}$ is an extendible metric which is flat around $p$. Then there exists a neighborhood $U$ of $p$ for which $\mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(M)$ is generic in $\mathcal{R}_{U,g_{\mathrm{flat}}}(M)$.*

Theorem 2.4 will follow from Theorems 4.1 and 7.1 below.

2.5. **The relation to the ADM mass.** Let $(M, g)$ be a compact spin manifold of dimension $n \geq 3$. Assume that $g$ is flat in a neighborhood $U$ of a point $p \in M$. The *conformal Laplacian* is then defined by

$$L^g := \frac{4(n-1)}{n-2}\Delta^g + \mathrm{scal}^g,$$

where $\Delta^g$ is the non-negative Laplacian and where $\mathrm{scal}^g$ is the scalar curvature of the metric $g$. As for the Dirac operator $D^g$, we say that a function $H^g \in L^1(M) \cap C^\infty(M \setminus \{p\})$ is the *Green's function* for $L^g$ if

$$L^g H^g = \delta_p$$

in the sense of distributions. Assume that the metric $g$ is conformal to a metric with positive scalar curvature. Then it is well known (see for instance [17]) that the Green's function $H^g$ of $L^g$ exists, is positive everywhere and has the following expansion at $p$:

$$H^g(x) = \frac{1}{4(n-1)\omega_{n-1}\, d^g(x,p)^{n-2}} + A^g + o(x),$$

where $A^g \in \mathbb{R}$ and $o(x)$ is a smooth function with $o(p) = 0$.

Set $\widetilde{M} = M \setminus \{p\}$ and $\widetilde{g} = H^{\frac{4}{n-2}} g$. Schoen [19] observed that the complete non-compact manifold $(\widetilde{M}, \widetilde{g})$ is asymptotically flat and its ADM mass is $a_n A^g$, where $a_n > 0$ depends only on $n$. We recall that an asymptotically flat manifold, if interpreted as a time symmetric spacelike hypersurface of a Lorentzian manifold, is obtained by considering an isolated system at a fixed time in general relativity. The ADM mass gives the total energy of this system. With this remark, the number $A^g$ is often called the mass of the compact manifold $(M, g)$. By analogy, the operator $\alpha^g(p)$, which is by construction the spin analog of $A^g$, is called the "mass endomorphism" of $(M, g)$ at $p$. We will also see in Subsection 2.6 that the mass endomorphism plays the same role as the number $A^g$ in a Dirac operator version of the Yamabe problem.

2.6. **Conclusions of non-zero mass.** In this Subsection we will summarize why we are interested in metrics with non-zero mass endomorphism.

Let $(M, g)$ be a compact Riemannian spin manifold of dimension $n \geq 2$. For a metric $\widetilde{g}$ in the conformal class $[g]$ of $g$, let $\lambda_1(\widetilde{g})$ be the eigenvalue of the Dirac operator $D^g$ with the smallest absolute value (it may be either positive or negative). We define

$$\lambda^+_{\min}(M, [g]) = \inf_{\widetilde{g} \in [g]} |\lambda_1(\widetilde{g})| \mathrm{Vol}^{\widetilde{g}}(M)^{1/n}.$$

For this conformal invariant $\lambda^+_{\min}(M, [g])$ it was proven in [1, 2] and [6] that

$$0 < \lambda^+_{\min}(M, [g]) \leq \lambda^+_{\min}(\mathbb{S}^n) = \frac{n}{2}\, \omega_n^{1/n}.$$

The strict inequality

$$\lambda^+_{\min}(M, [g]) < \frac{n}{2} \omega_n^{1/n} \tag{1}$$

has several applications, see [3, 6, 7]:

- Inequality (1) implies that the invariant $\lambda^+_{\min}(M, [g])$ is attained by a generalized metric, that is, a metric of the form $|f|^{2/(n-1)} g$ where $f \in C^2(M)$ can have some zeros;
- Inequality (1) gives a solution of a conformally invariant partial differential equation which can be read as a nonlinear eigenvalue equation for the Dirac operator, a type of Yamabe problem for the Dirac operator;
- using Hijazi's inequality [14] one obtains a solution of the standard Yamabe problem which consists of finding a metric with constant scalar curvature in the conformal class of $g$ in the case of $n \geq 3$.

The first two applications can for several reasons be interpreted as a spin analog of the Yamabe problem, see [1]. The third application says that a non-zero mass endomorphism can be used in the Yamabe problem instead of the positivity of the mass $A^g$ defined in Subsection 2.5.

Now, let us reconnect to the subject of this paper. In [7], we prove that a non-zero mass endomorphism implies Inequality (1). In particular we see with Theorem 2.4 that Inequality (1) holds for generic metric in $\mathcal{R}_{U,g_{\mathrm{flat}}}(M)$. As a consequence, for generic metrics in $\mathcal{R}_{U,g_{\mathrm{flat}}}(M)$, we have all the applications stated above.

This can be compared to the Yamabe problem: Schoen proved that the positivity of the number $A^g$, that is the mass of $(M,g)$ defined in Subsection 2.5, implies a solution of the standard Yamabe problem. The positive mass theorem implies that $A^g \geq 0$. Hence, we get a solution of the Yamabe problem as soon as $A^g \neq 0$. In particular, the mass endomorphism plays the same role in the Yamabe problem for the Dirac operator as the ADM mass in the classical Yamabe problem.

2.7. **Overview of the paper.** We here give a short overview of the paper. In Section 3 we introduce notation and collect basic facts concerning spinors and Dirac operators. In Section 4 we explain how to find one metric with non-zero mass endomorphism on a given manifold, this uses the results of the following two sections. In Section 5 we show that under certain assumptions the mass endomorphism tends to infinity when the Riemannian metric varies and approaches a metric with harmonic spinors. In Section 6 we show that the property of non-zero mass endomorphism can be preserved under surgery on the underlying manifold. Finally, in Section 7 we use analytic perturbation techniques to show that the existence of one metric with non-zero mass endomorphism implies that a generic metric has this property.

## 3. Notations and preliminaries

3.1. **Notation and some basic facts.** In this article we use the following notations for balls and spheres: $B^k(R) := \{x \in \mathbb{R}^k \,|\, \|x\| < R\}$, $B^k := B^k(1)$, $S^k(R) := \{x \in \mathbb{R}^k \,|\, \|x\| = R\}$, $S^k := S^k(1)$.

As background for basic facts on spinors and Dirac operators we refer to [16] and [11]. For the convenience of the reader we summarize here a few definitions and facts. On a compact Riemannian spin manifold $(M,g)$ one defines the Dirac operator $D^g$ acting on sections of the spinor bundle. The Dirac operator is essentially self-adjoint and extends to a self-adjoint operator $H^1 \to L^2$ where $H^1$ is the space of $L^2$-spinors whose first derivative is $L^2$ as well, and $L^2$ is the space of square integrable spinors. A smooth spinor is called *harmonic*, if it is in the kernel of the Dirac operator $D^g$. Any $L^2$-spinor satisfying $D^g\varphi = 0$ in the weak sense, is already smooth, thus it is a harmonic spinor. If the kernel of $D^g$ is trivial, then the Dirac operator is invertible with a bounded inverse $L^2 \to H^1$. The inverse has an integral kernel called the Green's function of $D^g$. The Green's function of $D^g$ has already been used in Subsection 2.1 to define the mass endomorphism.

3.2. **Comparing spinors for different metrics.** Let $g$ and $h$ be Riemannian metrics on the spin manifold $M$. The goal of this section is to recall how spinors on $(M,g)$ are identified with spinors on $(M,h)$ using the method of Bourguignon and Gauduchon [10], see also [5].

Given the metrics $g$ and $h$ there exists a unique bundle endomorphism $a^g_h$ of $TM$ which satisfies $g(a^g_h X, Y) = h(X,Y)$ for all $X, Y \in TM$. It is $g$-self-adjoint and positive definite. Define $b^g_h := (a^g_h)^{-1/2}$, where $(a^g_h)^{1/2}$ is the unique positive pointwise square root of $a^g_h$. The map $b^g_h$ maps $g$-orthonormal frames to $h$-orthonormal frames and defines an $\mathrm{SO}(n)$-equivariant bundle morphism $b^g_h : \mathrm{SO}(M,g) \to \mathrm{SO}(M,h)$

of the principal bundles of orthonormal frames. The map $b^g_h$ lifts to a Spin($n$)-equivariant bundle morphism $\beta^g_h : \mathrm{Spin}(M,g) \to \mathrm{Spin}(M,h)$ of the corresponding spin structures. From this we obtain a homomorphism of vector bundles

$$\beta^g_h : \Sigma^g M \to \Sigma^h M \tag{2}$$

which is a fiberwise isometry with respect to the inner products on $\Sigma^g M$ and $\Sigma^h M$. We let the Dirac operator $D^h$ act on sections of $\Sigma^g M$ by defining

$$D^h_g := (\beta^g_h)^{-1} D^h \beta^g_h.$$

In [10, Thm. 20] an expression for $D^h_g$ is computed in terms of a local $g$-orthonormal frame $\{e_i\}_{i=1}^n$. The result is

$$D^h_g \varphi = \sum_{i=1}^n e_i \cdot \nabla^g_{b^g_h(e_i)} \varphi + \frac{1}{2} \sum_{i=1}^n e_i \cdot ((b^g_h)^{-1} \nabla^h_{b^g_h(e_i)} b^g_h - \nabla^g_{b^g_h(e_i)}) \cdot \varphi, \tag{3}$$

where for any vector field $X$ the operator $(b^g_h)^{-1} \nabla^h_X b^g_h - \nabla^g_X$ is $g$-antisymmetric and therefore considered as an element of the Clifford algebra. It follows that

$$D^h_g \varphi = D^g \varphi + A^h_g(\nabla^g \varphi) + B^h_g(\varphi), \tag{4}$$

where $A^h_g$ and $B^h_g$ are pointwise vector bundle maps whose pointwise norms are bounded by $C|h-g|_g$ and $C(|h-g|_g + |\nabla^g(h-g)|_g)$ respectively.

## 4. Finding one metric with non-vanishing mass endomorphism

The goal of this section is to prove the following Theorem.

**Theorem 4.1.** *Let $M$ be a compact connected spin manifold of dimension $n \geq 3$ and let $p \in M$. Assume that $\alpha(M) = 0$. Then there exists a neighborhood $U$ of $p$ and a flat metric $g_{\mathrm{flat}}$ on $U$ such that $\mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(M)$ is non-empty.*

*Proof.* We start by proving the theorem when the manifold is a torus. Consider the torus $T^n$ equipped with the Lie group spin structure for which the standard flat metric $g_0$ has a space of parallel spinors of maximal dimension. Choose $p \in T^n$ and let $U$ be a small open neighborhood of $p$. Further, let $g_{\mathrm{flat}}$ be the restriction of $g_0$ to $U$.

Since $n \geq 3$ we have that $\alpha(T^n) = 0$ so by [5] there is a metric $g_1$ on $T^n$ with invertible Dirac operator. The construction of $g_1$ is done through a sequence of surgeries which starts with the disjoint union of $T^n$ and some other manifolds, and ends with the torus $T^n$. These surgeries can be arranged so that they do not change the open set $U$ in the initial $T^n$, so the resulting metric satisfies $g_1 = g_0$ on $U$, or $g_1 \in \mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(T^n)$.

Define the family of metrics $g_t := t g_1 + (1-t) g_0$. Since the eigenvalues of $D^{g_t}$ depend analytically on $t$ it follows that $D^{g_t}$ is invertible except for isolated values of $t$, it follows that $g_t \in \mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(T^n)$ except for isolated values of $t$. Choose a sequence $t_k \to 0$ for which $g_{t_k} \in \mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(T^n)$, we can then apply Theorem 5.1 below to the sequence $g_{t_k}$ converging to $g_0$ and conclude that $g_{t_k} \in \mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(T^n)$ for $k$ large enough. In particular $\mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(T^n)$ is not empty, and we choose a metric $h_0$ from this set.

Now let $M$ be a manifold of dimension $n$ as in the theorem. Since $\alpha(M) = 0$ we know that there is a metric $g$ on $M$ with invertible Dirac operator. We consider the disjoint union

$$M_0 = T^n \amalg (-T^n) \amalg M.$$

Here $-T^n$ denotes $T^n$ with the opposite orientation, so that $T^n \amalg (-T^n)$ is a spin boundary and $M_0$ is spin bordant to $M$. Since $M$ is connected it follows that $M$ can be obtained from $M_0$ by a sequence of surgeries of codimension 2 and higher, see [5, Proposition 4.3]. Again, these surgeries can be arranged to miss the open set $U$ in the first $T^n$. We equip $M_0$ with the Riemannian metric $h_0 \amalg h_0 \amalg g \in \mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(T^n \amalg (-T^n) \amalg M)$ and when we use Theorem 6.1 below for the sequence of surgeries we end up with a metric $g' \in \mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(M)$.

Finally, the point $p \in M$ we end up with after the sequence of surgeries might of course not be equal to the point $p$ in the assumptions of the theorem. If we set this right by a diffeomorphism we have proved that $\mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(M)$ is non-empty. $\square$

Note that this proof does not work in dimension 2. Indeed, we strongly use that the $\alpha$-genus of the torus $T^n$ vanishes. This fact is only true in dimension $n \geq 3$. If the flat torus $T^2$ is equipped with the Lie group spin structure with two parallel spinors, then $\alpha(T^2) = 1$. By the way, it is proven in [7] that the mass endomorphism always vanishes in dimension 2.

## 5. Mass endomorphism of metrics close to a metric with harmonic spinors

Finding examples of metrics with non-zero mass endomorphism seems to be a difficult issue. The only explicit examples we have until now are the projective spaces $\mathbb{R}P^n$, $n \equiv 3 \mod 4$, equipped with its standard metric, see [7]. The goal of this section is to show that metrics $g \in \mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(M)$ sufficiently close to a metric $h \in \mathcal{R}_{p,U,g_{\mathrm{flat}}} \setminus \mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(M)$ will under some additional assumptions provide such examples. This is the object of Theorem 5.1 below, which in our mind has an interest independently of the application to Theorem 2.4.

**Theorem 5.1.** *Let $U$ be a neighborhood of $p \in M$. Assume that $h \in \mathcal{R}_{U,g_{\mathrm{flat}}}(M)$ has $\ker D^h \neq \{0\}$. Further assume that the evaluation map of harmonic spinors at $p$,*

$$\ker D^h \ni \psi \mapsto \psi(p) \in \Sigma^h_p M,$$

*is injective. Set $m := \dim \ker D^h$ Let $g_k \in \mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(M)$, $k = 1, 2, \ldots,$ be a family of metrics on $M$ converging to $h$ in the $C^1$-topology.*

*Then the mass endomorphism $\alpha^{g_k}$ at $p$ has at least $m$ eigenvalues tending to $\infty$ as $k \to \infty$. In particular, $g_k \in \mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(M)$ for large $k$.*

The proof of this theorem is inspired by the work of Beig and O' Murchadha [9]. In the hypothesis of Theorem 5.1, the injectivity of the evaluation map $\ker D^h \ni \psi \mapsto \psi(p) \in \Sigma^h_p M$, is quite restrictive: it is fulfilled for instance when the space of harmonic spinors is 1-dimensional if $p$ is not a zero of the harmonic spinor. In Theorem 4.1 we applied the result to the flat torus $T^n$.

*Proof.* For the proof we choose a non-zero $\psi \in \ker D^h$. Set $\psi_p := \psi(p) \in \Sigma^h_p M$, by assumption we have $\psi_p \neq 0$. We will show that $\alpha^{g_k}(\psi_p)$ tends to infinity.

Let $G_k$ be the Green's function of $D^{g_k}$ associated to $\psi_p$, that is $G_k$ is a distributional solution of

$$D^{g_k} G_k = \delta_p \psi_p.$$

In coordinates around $p$ we write (compare Proposition 2.1)

$$G_k = -\eta \frac{x}{\omega_{n-1} r^n} \cdot \psi_p + v^{g_k}(\psi_p). \tag{5}$$

Here $\eta$ is a cutoff function which is equal to 1 near $p$ and has support in $U$. We shorten notation by writing $v_k$ for the spinor field $v^{g_k}(\psi_p)$.

**Step 1.** *We show that there are $p_k \in M$ for which $|v_k(p_k)| \to \infty$.* Let the smooth function $\Omega : M \setminus \{p\} \to (0, 1]$ satisfy

$$\Omega(x) = \begin{cases} r(x) & \text{if } x \in B_p(\varepsilon), \\ 1 & \text{if } x \in M \setminus B_p(2\varepsilon) \ . \end{cases}$$

Note that $\Omega$ does not depend on $k$. We have

$$\begin{aligned} 0 < |\psi_p|^2 &= \int_M \langle G_k, D^{g_k}\psi \rangle \, dv^{g_k} \\ &= \int_M \frac{1}{\Omega^{n-1}} \langle \Omega^{n-1} G_k, D^{g_k}\psi \rangle \, dv^{g_k} \\ &\leq \int_M \frac{1}{\Omega^{n-1}} \, dv^{g_k} \|\Omega^{n-1} G_k\|_\infty \|D^{g_k}\psi\|_\infty. \end{aligned}$$

As the integral is bounded and the last factor tends to zero as $k \to \infty$, we conclude that

$$\lim_{k\to\infty} \|\Omega^{n-1} G_k\|_\infty = \infty.$$

Let $p_k$ be points for which

$$|\Omega^{n-1}(p_k) G_k(p_k)| = \|\Omega^{n-1} G_k\|_\infty.$$

Then

$$\Omega^{n-1}(p_k) G_k(p_k) = \Omega^{n-1}(p_k) \left( -\eta \frac{x}{\omega_{n-1} r^n} \cdot \psi_0 \right)(p_k) + \Omega^{n-1}(p_k) v_k(p_k),$$

here the first term on the right hand side is bounded so the second term must tend to infinity. Since $|\Omega^{n-1}(p_k) v_k(p_k)| \leq |v_k(p_k)|$ we conclude that $|v_k(p_k)| \to \infty$ as $k \to \infty$, and Step 1 is proven.

To the spinor $v_k$ which is a section of $\Sigma^{g_k} M$ the map $\beta_h^{g_k}$ described in (2) associates a section $w_k := \beta_h^{g_k} v_k$ in the spinor bundle $\Sigma^h M$. We decompose this section as

$$w_k = a_k \varphi_k + w_k^\perp$$

where $\varphi_k \in \ker D^h$ is normalized to have $\|\varphi_k\|_{L^p(\Sigma^h M)} = 1$ , $a_k \in \mathbb{R}$, and $w_k^\perp$ is orthogonal to $\ker D^h$. We choose $p$ large enough so that $H_1^p(\Sigma^h M)$ embeds into $C^0(\Sigma^h M)$.

**Step 2.** *We show that $|a_k| \to \infty$.* For a contradiction assume that the sequence $|a_k|$ is bounded. From (5) it follows that $D^{g_k} v_k = \mathrm{grad}\eta \cdot \frac{x}{\omega_{n-1} r^n} \cdot \psi_p$. This together with the properties of $\beta_h^{g_k}$ gives

$$
\begin{aligned}
\|w_k^\perp\|_{H_1^p} &\leq C\|D^h w_k^\perp\|_{L^p} \\
&= C\|D^h w_k\|_{L^p} \\
&= C\|(\beta_h^{g_k})^{-1} D^h \beta_h^{g_k} v_k\|_{L^p} \\
&= C\|D_{g_k}^h v_k\|_{L^p} \\
&\leq C\|D^{g_k} v_k\|_{L^p} + C\|A_{g_k}^h(\nabla^{g_k} v_k) + B_{g_k}^h(v_k)\|_{L^p} \\
&\leq C\|\mathrm{grad}\eta \cdot \frac{x}{\omega_{n-1} r^n} \cdot \psi_p\|_{L^p} + C\varepsilon_k \|w_k\|_{H_1^p},
\end{aligned}
\tag{6}
$$

here the first term is bounded and $\varepsilon_k \to 0$ by our assumption that $g_k \to h$ in the $C^1$-topology. By assumption we also have

$$
\begin{aligned}
\|w_k\|_{H_1^p} &\leq \|a_k \varphi_k\|_{H_1^p} + \|w_k^\perp\|_{H_1^p} \\
&\leq C + \|w_k^\perp\|_{H_1^p}.
\end{aligned}
$$

Together this gives

$$\|w_k^\perp\|_{H_1^p} \leq C + C\varepsilon_k + C\varepsilon_k \|w_k^\perp\|_{H_1^p},$$

so $\|w_k^\perp\|_{H_1^p}$ is bounded. We conclude that $\|w_k^\perp\|_{C^0}$ is bounded, and the assumption that $|a_k|$ is bounded then tells us that $\|w_k\|_{C^0} = \|v_k\|_{C^0}$ is bounded. This contradicts Step 1, so we have proved Step 2.

**Step 3.** *Conclusion.* Set $\omega_k := a_k^{-1} w_k$ and $\omega_k^\perp := a_k^{-1} w_k^\perp$ so that

$$\omega_k = \varphi_k + \omega_k^\perp.$$

Then (6) tells us that

$$\|\omega_k^\perp\|_{H_1^p} \leq C a_k^{-1} \|\mathrm{grad}\eta \cdot \frac{x}{\omega_{n-1} r^n} \cdot \psi_0\|_{L^p} + C\varepsilon_k \|\omega_k\|_{H_1^p},$$

where the first term now tends to zero. Since the $\varphi_k$ are in $\ker D^h$ and they are normalized in $L^p(\Sigma^h M)$ it follows that they are bounded in $H_1^p(\Sigma^h M)$. From this we get

$$
\begin{aligned}
\|\omega_k\|_{H_1^p} &\leq \|\varphi_k\|_{H_1^p} + \|\omega_k^\perp\|_{H_1^p} \\
&\leq C + \|\omega_k^\perp\|_{H_1^p}.
\end{aligned}
$$

It follows that

$$\|\omega_k^\perp\|_{H_1^p} \leq o(1) + C\varepsilon_k \|\omega_k^\perp\|_{H_1^p}$$

so $\|\omega_k^\perp\|_{H_1^p} \to 0$ and $\|\omega_k^\perp\|_{C^0} \to 0$. Finally we have

$$
\begin{aligned}
|\alpha^{g_k}(\psi_p)| &= |v_k(p)| \\
&= |w_k(p)| \\
&= a_k |\omega_k(p)| \\
&\geq a_k(|\varphi_k(p)| - |\omega_k^\perp(p)|) \\
&= a_k(|\varphi_k(p)| + o(1)).
\end{aligned}
$$

By our assumption that the evaluation map of harmonic spinors at $p$ is injective we know that $|\varphi_k(p)|$ cannot tend to zero, so from Step 2 we conclude that $|\alpha^{g_k}(\psi_p)| \to \infty$. This finishes the proof of Step 3 and the theorem. ∎

## 6. Surgery and non-zero mass endomorphism

Let $\widehat{M}$ be obtained from $M$ by surgery of codimension at least 2. We assume that $p \in M$ is not hit by the surgery, so we have $p \in \widehat{M}$. As before $\mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(M)$ denotes the metrics with invertible Dirac operator on $M$ which coincide with the flat metric $g_{\mathrm{flat}}$ on $U$ and whose mass endomorphism at $p$ is not zero. The goal of this section is to prove that $\mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(M) \neq \emptyset$ implies $\mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(\widehat{M}) \neq \emptyset$.

We start with a manifold $M$ of dimension $n$ and a point $p \in M$. We will perform a surgery of dimension $k \in \{0, \cdots n-2\}$ on $M$. For this construction, we follow the beginning of Section 3 in [5] and use the same notation. So, we assume that we have an embedding $i : S^k \to M$ with a trivialization of the normal bundle of $S := i(S^k)$ in $M$, which thus can be identified with $S^k \times \mathbb{R}^{n-k}$. The normal exponential map then defines an embedding of a neighborhood of the zero section of the normal bundle of $S$, in other words for small $R > 0$ the normal exponential map defines a diffeomorphism $f$ from $S^k \times B^{n-k}(R)$ to an open neighborhood of $S$, and $f$ is an extension of $S^k \times \{0\} \to S^k \xrightarrow{i} M$. Furthermore, for sufficiently small $R > 0$, the distance from $f(x, y)$ to $S = f(S^k \times \{0\})$ is $|y|$.

As before we assume that $U$ is an open neighborhood of $p$, on which a flat extendible metric $g_{\mathrm{flat}}$ exists. We assume further that $p \notin S$, and by possibly restricting $U$ to a smaller open set, we can also assume that $\overline{U} \cap S = \emptyset$. Thus for small $R > 0$ one obtains

$$U \cap f(S^k \times \overline{B^{n-k}(R)}) = \emptyset.$$

As in Section 1 of [5] we define

$$\widehat{M} = \left(M \setminus f(S^k \times \overline{B^{n-k}(R)})\right) \cup \left(\overline{B^{k+1}} \times S^{n-k-1}\right) /\!\sim,$$

where $\sim$ identifies the boundary of $\overline{B^{k+1}} \times S^{n-k-1}$ with $f(S^k \times S^{n-k-1}(R))$ via the map $(x, y) \mapsto f(x, Ry)$. Our constructions are carried out such that $U$ is both a subset of $M$ and $\widehat{M}$.

The main result of this section is the following Theorem.

**Theorem 6.1.** *If $\mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(M) \neq \emptyset$, then $\mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(\widehat{M}) \neq \emptyset$.*

*Proof.* We assume the requirements for $p$, $U$, $f$ and $k$ stated at the beginning of this section, and let $g \in \mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(M)$. The goal is to construct a metric $\widehat{g} \in \mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(\widehat{M})$ following the constructions in [5].

Theorem 1.2 in [5] allows us to construct a metric $\widehat{g}'$ on $\widehat{M}$ with invertible Dirac operator. We recall the scheme of the proof of this theorem. As in the beginning of Section 3 of [5] we define open neighborhoods $U_S(r)$ by

$$U_S(r) := f(S^k \times B^{n-k}(r))$$

for small $r$. Then we construct a family of metrics $(g_\rho)_\rho$ satisfying $g_\rho = g$ on $M \setminus U_S(R_{\max})$ for some small number $R_{\max}$. This family of metrics is constructed in two steps. First, we use Proposition 3.2 in [5] to assume that $g$ has a product form in a neighborhood of $S$. Then, we do the construction of Section 3.2 in [5] to get $g_\rho$. Once these metrics $(g_\rho)$ are constructed, we proceed by contradiction. We take a sequence $(\rho_k)_{k\in\mathbb{N}}$ tending to 0 and we assume that $\ker(D^{g_{\rho_k}}) \neq 0$ for all $k$, that is

$$\forall k \in \mathbb{N}, \text{ there exists a harmonic spinor } \psi_k \neq 0 \text{ on } (\widehat{M}, g_{\rho_k}). \qquad (7)$$

By showing that $\lim_{k\to\infty}\psi_k$ converges in a weak sense to a non-zero limit spinor in $\ker D^g$, we will obtain a contradiction. So the metric $\widehat{g'} := g_\rho$ satisfies the requirements of Theorem 1.2 in [5] as soon as $\rho$ is small enough.

This proof actually allows us to require an additional property for the metrics $g_\delta$, and make weaker assumptions on the spinors $\psi_k$.

- The number $R_{\max}$ in the proof can be chosen arbitrarily small. So set $\delta = R_{\max}$ and choose $\rho := \rho(\delta)$ small enough so that $g_\delta = g_\rho$ has an invertible Dirac operator. We obtain in this way a family of metrics $(g_\delta)_{\delta\in(0,\delta_0)}$ for some $\delta_0 > 0$ such that all $D^{g_\delta}$ are invertible and such that $g_\delta = g$ on $M \setminus U_S(\delta)$.
- Let now $(\delta_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers going to 0. We make the following assumption:

  $$\forall k \in \mathbb{N}, \text{ there exists a spinor } \psi_k \text{ on } (\widehat{M}, g_{\delta_k}) \text{ and a sequence } \lambda_k \text{ converging to 0 such that } D^{g_{\delta_k}}\psi_k = \lambda_k\psi_k.$$

  Working with these spinors instead of the ones given by assumption (7), the same contradiction is obtained. This proves that there is a uniform spectral gap for $(g_\delta)_{\delta\in(0,\delta_0/2)}$, or in other words that there exists a constant $C_0 > 0$ independent of $\delta \in (0, \delta_0/2)$ such that

  $$\mathrm{Spec} D^{g_\delta} \cap [-C_0, C_0] = \emptyset. \tag{8}$$

Now, we prove that the metric $\widehat{g} := g_\delta$ for $\delta$ small enough satisfies the requirements of Theorem 6.1. It is already clear that $D^{g_\delta}$ is invertible for $\delta$ small enough, and that $g_\delta$ is flat on $U$ for $\delta$ small enough. It remains to show that $\alpha_p^{g_\delta} \neq 0$ for $\delta$ small enough. For this purpose we show that $\alpha_p^{g_\delta} \to \alpha_p^g$ as $\delta \to 0$. Since we assume $\alpha_p^g \neq 0$ this gives the desired result.

So let us prove this fact. First, choose $\psi_0 \in \Sigma_p^g(M) = \Sigma_p^{g_\delta}(M)$. To simplify the notation, set $\gamma := G^g\psi_0$ and $\gamma_\delta := G^{g_\delta}\psi_0$. The proof will be complete if we prove that

$$\lim_{\delta\to 0} \gamma(p) - \gamma_\delta(p) = 0. \tag{9}$$

Note that the spinor $\gamma - \gamma_\delta$, defined on $M \setminus (\{p\} \cup U_S(\delta))$, is smooth and extends smoothly to $p$. Indeed, it is equal on $U$ to $v_p^g(x)\psi_0 - v_p^{g_\delta}(x)\psi_0$ (with the notations of Proposition 2.1 and Definition 2.2). Let $\eta_\delta \in C^\infty(\widehat{M})$, $0 \le \eta_\delta \le 1$ be a cut-off function such that $\eta_\delta = 1$ on $M \setminus U_S(3\delta)$ and $\eta_\delta = 0$ on $U_S(2\delta)$. Since on $\mathrm{supp}(\eta_\delta) \subset \widehat{M} \setminus U_S(2\delta) = M \setminus U_S(2\delta)$ we have $g_\delta = g$ we may assume that

$$|d\eta_\delta|_g = |d\eta_\delta|_{g_\delta} \le \frac{2}{\delta}. \tag{10}$$

From Equation (8), we have

$$C_0^2 \le \frac{\int_{\widehat{M}} |D^{g_\delta}\varphi_\delta|^2_{g_\delta} dv^{g_\delta}}{\int_{\widehat{M}} |\varphi_\delta|^2_{g_\delta} dv^{g_\delta}}$$

for all smooth non-zero spinors $\varphi_\delta$ on $(\widehat{M}, g_\delta)$. We evaluate this quotient for $\varphi_\delta := \eta_\delta\gamma - \gamma_\delta$. Note that $\varphi_\delta$ is well defined on $(\widehat{M}, g_\delta)$ and smooth since $\gamma$ is well defined on $\mathrm{supp}(\eta_\delta)$. Since $\gamma$ and $\gamma_\delta$ are harmonic, we have $D\varphi_\delta = d\eta_\delta \cdot \gamma$, and since $g_\delta = g$

on $\mathrm{supp}(\eta_\delta)$, we get from Equation (10) that

$$\begin{aligned}\int_{\widehat{M}} |D^{g_\delta}\varphi_\delta|^2_{g_\delta} dv^{g_\delta} &= \int_{\widehat{M}} |d\eta|^2_g |\gamma|^2_g dv^g \\ &\leq \frac{4}{\delta^2} \sup_{x\in U_S(3\delta_0)} \left(|\gamma(x)|^2\right) \mathrm{Vol}^g\left(U_S(3\delta)\setminus U_S(2\delta)\right).\end{aligned}$$

We have that $\mathrm{Vol}^g\left(U_S(3\delta)\setminus U_S(2\delta)\right) \leq C\delta^{n-k}$ where we used the convention (used throughout this proof) that $C$ is a positive constant independent of $\delta$. Since $k \leq n-2$, this leads to

$$\int_{\widehat{M}} |D^{g_\delta}\varphi_\delta|^2_{g_\delta} dv^{g_\delta} \leq C.$$

Since $\eta_\delta = 1$ on $M \setminus U_S(3\delta)$ and since $g_\delta = g$ on this set, it follows that

$$\int_{M\setminus U_S(3\delta)} |\varphi_\delta|^2_{g_\delta} dv^{g_\delta} \leq C. \tag{11}$$

Now, we proceed as in step 2 of the proof of Theorem 1.2 in [5]. Let $Z > 0$ be a large integer. By (11) the set $\{\varphi_\delta\}_{\delta>0}$ is bounded in $L^2(M \setminus U_S(1/Z))$. By Lemma 2.2 in [5] it follows that $\{\varphi_\delta\}_{\delta>0}$ is bounded in $C^{1,\alpha}(M\setminus U_S(2/Z))$ for all $\alpha$. We apply Ascoli's Theorem and conclude there is a subsequence $(\varphi_{\delta_k})$ of $\{\varphi_\delta\}_{\delta>0}$ which converges in $C^1(M\setminus U_S(2/Z))$ to a spinor $\Phi_0$. Similarly we construct further and further subsequences of $(\varphi_{\delta_k})$ converging to $\Phi_i$ in $C^1(M\setminus U_S(2/(Z+i)))$. Taking a diagonal subsequence of these subsequences, we obtain a subsequence $(\varphi_{\delta_k})$ which converges in $C^1_{\mathrm{loc}}(M\setminus S)$ to a spinor $\Phi$. As $\varphi_\delta$ is $D^g$-harmonic on $(M\setminus U_S(3\delta))$ the $C^1_{\mathrm{loc}}(M\setminus S)$-convergence implies that $D^g\Phi = 0$ on $M \setminus S$. With (11) we conclude that $\Phi \in L^2(M)$. Thus $\Phi$ is $L^2$ and smooth on $M \setminus S$. The equation $D^g\Phi = 0$ holds on $M \setminus S$. We now apply Lemmas 2.1 and 2.4 of [5] and conclude that $\Phi$ is smooth on $(M, g)$ and $D\Phi = 0$ on $M$. Since $\ker D_0 = 0$, we get that $\Phi \equiv 0$ and in particular $\Phi(p) = 0$. This implies Equation (9). □

## 7. From existence to genericity

The goal of this section is to prove the following Theorem.

**Theorem 7.1.** *Let $M$ be a compact spin manifold of dimension $n$, $n \geq 3$, let $p \in M$ and let $U$ be a neighborhood of $p$. If $\mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(M)$ is non-empty then it is generic in $\mathcal{R}_{U,g_{\mathrm{flat}}}(M)$.*

### 7.1. Continuity of the mass endomorphism.

The goal of this subsection is to prove that the mass endomorphism depends continuously on $g$ in the $C^1$-topology.

**Proposition 7.2.** *Equip $\mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(M)$ with the $C^1$-norm. Then the map*

$$\mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(M) \ni g \mapsto \alpha^g \in \mathrm{End}(\Sigma_p M)$$

*is continuous.*

It follows that $\mathcal{R}^{\neq 0}_{p,U,g_{\mathrm{flat}}}(M)$ is open in $\mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(M)$ and thus in $\mathcal{R}_{U,g_{\mathrm{flat}}}(M)$.

*Proof.* Let $(g_k)_{k\in\mathbb{N}}$ be a family of metrics in $\mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(M)$ such that $g_k \to g$ in the $C^1$-topology. For each $k$ the operator

$$D^{g_k}_g = (\beta^g_{g_k})^{-1} D^{g_k} \beta^g_{g_k}$$

is invertible. We define

$$P_k := D_g^{g_k} - D^g.$$

Further, let $G^{g_k}$ and $G^g$ be the Green's functions of $D^{g_k}$ and $D^g$. We define

$$Q_k := (\beta_{g_k}^g)^{-1} G^{g_k} \beta_{g_k}^g - G^g.$$

Let $\psi \in \Sigma_p M$. Using the equation (5) for $G^{g_k}$ and for $G^g$ and using the fact that $g_k|_U = g|_U = g_{\mathrm{flat}}$ we find that

$$Q_k\psi = (\beta_{g_k}^g)^{-1} v^{g_k} \beta_{g_k}^g \psi - v^g \psi.$$

Therefore $Q_k\psi$ has a smooth continuation to all of $M$. The equation $D^{g_k} G^{g_k} = D^g G^g = \delta_p \operatorname{Id}_{\Sigma_p M}$ then tells us that

$$Q_k = -(D_g^{g_k})^{-1} P_k G^g.$$

$(G^g\psi)(x)$ becomes singular as $x \to p$. However we may take a smooth function $\eta$ which is equal to 1 near $p$ and has support in $U$ and since $g_k|_U = g|_U = g_{\mathrm{flat}}$ we obtain

$$P_k(\eta G^g \psi) = D^g(\eta G^g\psi) - D^g(\eta G^g \psi) = 0.$$

It follows that $P_k G^g\psi = P_k(1-\eta)G^g\psi$, where $(1-\eta)G^g\psi$ is smooth on all of $M$. From (4) it follows that the sequence $(D_g^{g_k})_{k\in\mathbb{N}}$ converges to $D^g$ with respect to the norm of bounded linear operators from $C^1(\Sigma^g M)$ to $C^0(\Sigma^g M)$. Therefore $\|P_k G^g\psi\|_{C^0} \to 0$ as $k \to \infty$. Then it follows from [15, Thm. IV-1.16] that $((D_g^{g_k})^{-1})_{k\in\mathbb{N}}$ converges to $(D^g)^{-1}$ with respect to the norm of bounded linear operators from $C^0(\Sigma^g M)$ to $C^1(\Sigma^g M)$. Therefore $\|Q_k\psi\|_{C^1} \to 0$ as $k \to \infty$. Evaluating $Q_k$ at $p$ yields $\alpha^{g_k} - \alpha^g$. Thus the statement of the Proposition follows. □

7.2. **Analyticity of the mass endomorphism.** In this section $M$ is a closed spin manifold.

**Definition 7.3.** Let $\varepsilon > 0$. A family $(g_t)_{t\in(-\varepsilon,\varepsilon)}$ of Riemannian metrics on $M$ is called real analytic if there exist sections $h_k$ of the bundle of symmetric bilinear forms on $M$, $k \in \mathbb{N}$, such that for all $t \in (-\varepsilon,\varepsilon)$ and for all $r \in \mathbb{N}$ we have $\|g_t - \sum_{k=0}^N t^k h_k\|_{C^r} \to 0$ as $N \to \infty$.

Let $r, s \in \mathbb{N}$. A holomorphic family $\varphi$ in $C^r(\Sigma^g M)$ is a map $\varphi\colon \Omega \to C^r(\Sigma^g M)$, where $\Omega$ is an open subset in $\mathbb{C}$ and $\varphi$ is differentiable in the norm $\|.\|_{C^r(\Sigma^g M)}$. A holomorphic family $P$ in the space of bounded operators $B(C^r(\Sigma^g M), C^s(\Sigma^g M))$ is a map $P\colon \Omega \to B(C^r(\Sigma^g M), C^s(\Sigma^g M))$, where $\Omega$ is an open subset in $\mathbb{C}$ and $P$ is differentiable in the operator norm $\|.\|_{B(C^r(\Sigma^g M),C^s(\Sigma^g M))}$. The terminology we use here is the same as in Kato's book [15]. In particular we may use that a family $P$: $\Omega \to B(C^r(\Sigma^g M), C^s(\Sigma^g M))$ is holomorphic if and only if it is weakly holomorphic, i.e. if and only if for every fixed $\varphi \in C^r(\Sigma^g M)$ the family $P\varphi\colon \Omega \to C^s(\Sigma^g M)$ is holomorphic. The restriction of a holomorphic family of spinors or operators to a real interval will be called a real-analytic family.

We first show that for every real-analytic family $(g_t)_{t\in(-\varepsilon,\varepsilon)}$ of Riemannian metrics on $M$ the family $(D_g^{g_t})_{t\in(-\varepsilon,\varepsilon)}$ of Dirac operators is real analytic. The authors of the article [10] state a similar result, however they consider the Dirac operator as a closed operator on the space of $L^2$-spinors.

**Lemma 7.4.** *Let $M$ be a closed spin manifold and let $(g_t)_{t\in(-\varepsilon,\varepsilon)}$ be a real analytic family of Riemannian metrics in $\mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(M)$. Then the family $(D_g^{g_t})_{t\in(-\varepsilon,\varepsilon)}$ is a real analytic family in $B(C^1(\Sigma^g M), C^0(\Sigma^g M))$.*

*Proof.* It is sufficient to show that for every fixed $\varphi \in C^1(\Sigma^g M)$ the family of spinors $(D_g^{g_t}\varphi)_{t\in(-\varepsilon,\varepsilon)}$ is real analytic. As in section 3.2 we define endomorphisms $a_{g_t}^g$ and $a_{h_k}^g$, $k \in \mathbb{N}$, of $TM$ such that for all $X$, $Y$ in $TM$ we have

$$g(a_{g_t}^g X, Y) = g_t(X,Y), \quad g(a_{h_k}^g X, Y) = h_k(X,Y).$$

Note that $a_{h_k}^g$ also exists if $h_k$ is not positive definite. Let $|.|$ be the norm on $\Sigma^g M$ induced by the inner product and let $\{e_i\}_{i=1}^n$ be a local $g$-orthonormal frame. Since $(g_t)_{t\in(-\varepsilon,\varepsilon)}$ is real analytic it follows that

$$\begin{aligned} &\sup_{X\in TM,\,|X|=1} |a_{g_t}^g X - \sum_{k=0}^N t^k a_{h_k}^g X| \\ = &\sup_{X\in TM,\,|X|=1} |\sum_{i=1}^n g(a_{g_t}^g X, e_i)e_i - \sum_{i=1}^n \sum_{k=0}^N t^k g(a_{h_k}^g X, e_i)e_i| \to 0, \quad N \to \infty \end{aligned}$$

for all $t \in (-\varepsilon, \varepsilon)$. From this calculation it follows that for every vector field $X$ of length 1 on $M$ the vector field $b_{g_t}^g X$ is also given by a convergent power series and the convergence is uniform in $X$. Furthermore for any vector fields $X$, $Y$ the vector field $\nabla_X^{g_t} Y$ is also given by a convergent power series as can be seen in local coordinates. The assertion now follows from the formula (3) for $D_g^{g_t}$. □

**Proposition 7.5.** *If $(g_t)_{t\in(-\varepsilon,\varepsilon)}$ is a real-analytic family of metrics in $\mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(M)$, then $\alpha^{g_t}$ is also real-analytic.*

*Proof.* It is sufficient to show that for every $\psi \in \Sigma_p M$ the family of spinors

$$Q_t\psi := (\beta_{g_t}^g)^{-1} G^{g_t} \beta_{g_t}^g \psi - G^g \psi$$

is a real analytic family in $C^0(\Sigma^g M)$. As above we define $P_t := D_g^{g_t} - D^g$ and we obtain

$$Q_t = -(D_g^{g_t})^{-1} P_t G^g.$$

By the previous lemma the family of operators $(D_g^{g_t})_{t\in(-\varepsilon,\varepsilon)}$ is real analytic. It follows from [15, VII-§1.1] that the family of operators $((D_g^{g_t})^{-1})_{t\in(-\varepsilon,\varepsilon)}$ is also real analytic. As in the proof of Proposition 7.2 one concludes that for every $t$ the spinor $P_t G^g \psi$ is smooth on all of $M$. Thus the family $(P_t G^g \psi)_{t\in(-\varepsilon,\varepsilon)}$ is a real analytic family in $C^1(\Sigma^g M)$ and thus the family $(Q_t\psi)_{t\in(-\varepsilon,\varepsilon)}$ is a real analytic family in $C^0(\Sigma^g M)$. □

Consider a real analytic family $(g_t)_{t\in(a,b)}$ of Riemannian metrics on $M$. By unique continuation we immediately see: If there is a $t_0 \in (a,b)$ with $\alpha^{g_{t_0}} \neq 0$, then the set

$$S := \{t \in (a,b) \,|\, \alpha^{g_t} = 0\}$$

is a discrete subset of $(a,b)$.

Two metrics in the same connected component of $\mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}(M)$ can be joined by a piecewise real-analytic path of metrics. It follows that if a connected component of $\mathcal{R}^{\mathrm{inv}}_{U,g_{\mathrm{flat}}}$ contains at least one metric with non-zero mass endomorphism, then the metrics with non-zero mass endomorphism are dense in this component. In order to obtain Theorem 7.1, we still have to discuss families $(g_t)_{t\in(a,b)}$ where $D^{g_t}$ is not invertible for some $t$. As the mass endomorphism is not defined for these $t$, we complexify the parameter $t$ and pass around the metric with non invertible $D^{g_t}$ in the imaginary direction. This is discussed in the following subsection.

7.3. **Analytic continuation in the imaginary direction.** Again let $(g_t)_{t\in(a,b)}$ be a real-analytic family of metrics. We assume $g_t \in \mathcal{R}_{U,g_{\mathrm{flat}}}(M)$ for any $t \in (a,b)$, but we do not assume that all $D^{g_t}$ are invertible. Because of the real-analyticity of $D^{g_t}_g$, the family can be extended to a complex-analytic family of operators defined for $t$ in an open subset $U \supset (a,b)$ of $\mathbb{C}$. In this complexification the operators $D^{g_t}_g$ will no longer be self-adjoint, instead we have $(D^{g_t}_g)^* = D^{g_{\overline{t}}}_g$.

As the set of invertible operators is open, we can assume without loss of generality that $D^{g_t}_g$ is invertible on $U \setminus (a,b)$. In other words we assume that

$$T := \{t \in U \mid D^{g_t}_g \text{ is not invertible}\}$$

is contained in $(a,b)$.

The arguments from above also yield that $t \mapsto \alpha^{g_t}$ is a holomorphic function on $U \setminus T$. As $U \setminus T$ is connected, unique continuation implies the following Proposition.

**Proposition 7.6.** *If the mass endomorphism $\alpha^{g_{t_0}}$ is non-zero for any $t_0 \in (a,b) \setminus T$, then*

$$\{t \in (a,b) \setminus T \mid \alpha^{g_t} \neq 0\}$$

*is dense in $(a,b)$.*

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