ANALYSIS AND BOUNDARY VALUE PROBLEMS ON SINGULAR DOMAINS: AN APPROACH VIA BOUNDED GEOMETRY

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Abstract. We prove well-posedness and regularity results for elliptic boundary value problems on certain singular domains that are conformally equivalent to manifolds with boundary and bounded geometry. Our assumptions are satisfied by the domains with a smooth set of singular cuspidal points, and hence our results apply to the class of domains with isolated oscillating conical singularities. In particular, our results generalize the classical L^2-well-posedness result of Kondratiev for the Laplacian on domains with conical points. However, our domains and coefficients are too general to allow for singular function expansions of the solutions similar to the ones in Kondratiev's theory. The proofs are based on conformal changes of metric, on the differential geometry of manifolds with boundary and bounded geometry, and on our earlier geometric and analytic results on such manifolds.

Version française abrégée. Nous prouvons des résultats de solvabilité et régularité pour des systèmes satisfaisants la condition de Legendre forte avec conditions au bord mixtes de type Dirichlet-Neumann sur certains domaines singuliers. Notre classe de domaines singuliers contient la classe des domaines avec des singularités coniques isolées. Nos résultats généralisent ainsi le théorème d'isomorphisme de Kondratiev [17]. Dans la suite, M sera une variété riemannienne lisse à bord de dimension m et $E \to M$ sera un fibré vectoriel hermitien équipé d'une connexion. Pour nos résultats, nous allons aussi supposer que M ait une géométrie bornée. Soit a une forme sesquilinéaire lisse sur $T^*M \otimes E$ et $P_a \colon H^1(M;E) \to H^1(M;E)^*$ défini par la formule $\langle P_a u, v\rangle := \int_M a(\nabla u, \nabla v) d\,\mathrm{vol}_g$, pour $u, v \in H^1(M;E)$. Nos espaces de fonctions seront les espaces de Sobolev pondérés de type Kondratiev, voir Équation (1). Nous supposons donée une partition $\partial M = \partial_0 M \sqcup \partial_1 M$ du bord en deux sous-ensembles disjoints et ouverts, ainsi que des conditions au bord B_j d'ordre j, B_j sur $\partial_j M$. Nos résultats sont alors:

(i) P satisfait la régularité dans les espaces pondérés $f\mathcal{K}^{\ell,2}_{(\rho)}$ de l'équation (1) si, et seulement si les conditions au bord $B = (B_0, B_1)$ satisfont la condition de régularité de Shapiro-Lopatinski uniforme. Ces conditions sont satisfaites pour les opérateurs satisfaisant les conditions de Legendre fortes avec des conditions au bord mixtes (Dirichlet/Neumann). On obtient en particulier des résultats de régularité pour l'opérateur de Laplace avec conditions au bord mixtes, Théorème 6.

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(ii) Si, en plus des conditions de (i), P satisfait une inégalité de Hardy-Poincaré, alors le problème au bord associé à P est également bien posé.

En principe, la classe des domaines à laquelle nos résultats s'appliquent est assez large, mais pour des raisons d'espace et afin de réduire au minimum les détails techniques, nous considérons dans cette note principalement les exemples de domaines cuspidaux et wedge. L'ensemble des points singuliers de ces domaines est une sous-variété lisse compacte.

Introduction. We prove well-posedness and regularity results for systems of partial differential equations satisfying the strong Legendre condition with mixed Dirichlet-Neumann boundary conditions on certain singular domains. Our class of singular domains includes the class of domains with isolated conical singularities and thus they generalize the classical well-posedness result of Kondratiev [17]. Unlike Kondratiev's theory, singular functions expansions are not possible in our setting.

Let us briefly state our main result. Here are first, our assumptions. Throughout this paper, (M, g) will be a smooth, m-dimensional Riemannian manifold with boundary and $E \to M$ will be a hermitian vector bundle with connection ∇ such that its curvature R^E and all its covariant derivatives $\nabla^j R^E$, $j \geq 1$, are bounded. For our results, we shall also assume that M has bounded geometry (a concept recalled below, see, however [4, 5, 7, 14] for the concepts not recalled in this paper). Let a be a smooth, sesquilinear form on $T^*M \otimes E$ and $P_a \colon H^1(M;E) \to H^1(M;E)^*$ be defined by the formula $\langle P_a u, v\rangle := \int_M a(\nabla u, \nabla v) d\,\mathrm{vol}_g$, for $u, v \in H^1(M;E)$. Recall that if a is uniformly positive definite, then a is said to satisfy the *strong Legendre condition*. If $P = P_a + Q$, where Q is of order ≤ 1, we shall say that P satisfies the strong Legendre condition if, and only if, a does. This implies that P is strongly elliptic. For scalar operators, the condition that P satisfies the strong Legendre condition is actually equivalent to P being uniformly strongly elliptic. A smooth function $f \colon M \to (0, \infty)$ will be called an *admissible weight* if $f^{-1}df$ has bounded covariant derivatives of all orders. Let f and ρ be admissible weights on M. If $g_0 := \rho^2 g$ and ∇_0 is the Levi-Civita connection associated to g_0, we can describe our function spaces as the following Kondratiev-type weighted Sobolev spaces

$$f\mathcal{K}^{\ell,p}_{(\rho)}(M, g_0; E) := \big\{\psi \mid \rho^j \nabla_0^j (f^{-1}\psi) \in L^p(M, g_0; T^*M^{\otimes j} \otimes E),\ (\forall)\, j \leq \ell\,\big\}. \quad (1)$$

In our applications and in some of our results, the weight ρ is bounded. For simplicity, we shall assume this throughout the paper. We also assume that we have a partition $\partial M = \partial_0 M \sqcup \partial_1 M$ of the boundary in two disjoint, open subsets. We assume that we are given boundary conditions B_j of order j, B_j on $\partial_j M$, satisfying the boundedness and smoothness conditions stated before Theorem 6. Our results are then as follows (for (M, g) with bounded geometry):

(i) P satisfies regularity in the weighted spaces $f\mathcal{K}^{\ell,2}_{(\rho)}$ of Equation (1) if, and only if $B = (B_0, B_1)$ satisfies the uniform Shapiro-Lopatinski regularity conditions. These conditions are satisfied for operators satisfying the strong Legendre conditions with mixed (Dirichlet/Neumann) boundary conditions. In particular, the Laplace operator satisfies regularity for mixed boundary conditions, Theorem 6.

(ii) If, in addition to the conditions of (i), P satisfies a Hardy-Poincaré inequality, then P also satisfies a well-posedness result. We provide several examples of how to obtain the Hardy-Poincaré inequality.

In principle, the class of domains to which our results apply is pretty large, but for reasons of space and in order to keep the technicalities to a minimum, we mostly consider the examples of canonical cuspidal and wedge domains introduced by H. Amann [1], whose definition is recalled below. The set of singular points of such a domain is smooth and compact (without corners). It is even a finite set for cuspidal domains. Some very general and nice results were obtained in [12, 18] for certain domains with isolated point cusp singularities. Our methods are quite different, relying heavily on differential geometry, and thus allowing us to treat a large class of domains. Moreover, our coefficients are less regular than the ones in the references, but we lose the Fredholm properties and the singular function expansions obtained in [12, 18]. Algebras of pseudodifferential operators on manifolds with cuspidal points were considered in [25]. The index problem on such manifolds was considered in [19]. We thank Herbert Amann for useful comments.

Manifolds with boundary and bounded geometry. In this paper, (M, g) will always be a smooth, m-dimensional Riemannian manifold with boundary and $E \to M$ will be an vector bundle with metric and metric preserving connection. A smooth function $f\colon M \to (0,\infty)$ will be called *a g-admissible weight* if $f^{-1}df$ has bounded covariant derivatives of all orders. We shall say that E has *totally bounded curvature* if its curvature and all of its covariant derivatives are bounded. We endow TM with the Levi-Civita connection ∇ associated to g. Recall that M is said to have *bounded geometry* if its injectivity radius $r_{\mathrm{inj}}(M) > 0$ is positive and if TM has *totally bounded curvature*. We assume from now on that E is complex and it has totally bounded curvature.

Let us consider a codimension one submanifold $H \subset M$ (i.e. hypersurface). Assume that H carries a globally defined unit normal vector field ν and let $\exp^{\perp}(x,t) := \exp^M_x(t\nu_x)$ be the exponential in the direction of the chosen unit normal vector. By II^H we denote the *second fundamental form* of H. The following two definitions are from [5].

Definition 1. Let $(\widehat{M}, \widehat{g})$ be a Riemannian manifold with bounded geometry. We say that $H \subset \widehat{M}$ is a *bounded geometry hypersurface* in M if it is a closed subset of M, if $\|\nabla^k \mathrm{II}^H\|_{L^\infty} < \infty$ for all $k \geq 0$, and if here is $r_\partial > 0$ such that $\exp^\perp\colon H \times (-r_\partial, r_\partial) \to \widehat{M}$ is a diffeomorphism onto its image.

Definition 2. A Riemannian manifold (M, g) with boundary is said to have *bounded geometry* if there is a Riemannian manifold $\widehat{M}$ with bounded geometry containing M as an open subset such that ∂M is a bounded geometry hypersurface in $\widehat{M}$.

Remark 3. In [1], Amann has introduced the class of "singular manifolds." A *singular manifold* (M, g_0, ρ) is a Riemannian manifold with boundary (M, g_0) together with a *singularity function* ρ satisfying suitable properties. In particular, $(M, \rho^{-2}g_0)$ is assumed to be a manifold with boundary and bounded geometry. Conversely, if (M, g) has bounded geometry and ρ is a g-admissible weight, then $(M, g_0 := \rho^2 g)$ is a singular manifold with singularity function ρ. In the boundaryless case, this was first noticed in [13] (see also [2].) The singularity function ρ is seen to be a g-admissible weight. For manifolds with boundary, this result follows from [5] or [15]. The results of [1] apply therefore to the setting of manifolds with boundary and bounded geometry endowed with an admissible weight. A triple (M, g, ρ) consisting of a manifold with boundary and bounded geometry and a bounded g-admissible weight ρ will be called an *Amann triple*.

Conformal changes of metric. If $h_1, h_2 \colon X \to (0,\infty)$, we shall write $h_1 \sim h_2$ if h_1/h_2 and h_2/h_1 are both bounded. Let g_0 be a second Riemannian metric on M, whose Levi-Civita connection is denoted ∇_0. Let $\rho, f \colon M \to (0,\infty)$ be measurable functions and $p \in [1,\infty]$. Recall then from Equation (1) the definition of the spaces $f\mathcal{K}^{\ell,p}_{(\rho)}(M, g_0; E)$, which reduce to the usual Sobolev spaces if $\rho, f \sim 1$. More precisely, if $g := \rho^{-2} g_0$ and if ρ is a g-admissible weight and f is continuous, then the weighted and classical spaces are related by

$$f\mathcal{K}^{\ell,p}_{(\rho)}(M, g_0; E) = f\rho^{-m/p} W^{\ell,p}(M, g; E), \quad 1 \le p \le \infty. \tag{2}$$

(see [1, 3, 6] and Remark 3). We drop the superscript p for $p = 2$: $\mathcal{K}^{\ell}_{(\rho)}(M, g_0; E) := \mathcal{K}^{\ell,2}_{(\rho)}(M, g_0; E)$ and so on. We assume from now on that $g = \rho^{-2} g_0$.

Example 4. A typical example is when $M \subset \mathbb{R}^m$ is the closed unit ball, g_0 is the euclidean metric, and $\rho = r^{\lambda}$, where r is the distance to the origin. Then $(M, g := \rho^{-2} g_0)$ has bounded geometry if, and only if, $\lambda \ge 1$. Moreover,

$$f \ := \begin{cases} e^{-(\frac{r}{\epsilon})^{-\epsilon}}, & \text{if } \lambda = 1 + \epsilon > 1, \\ r = \rho, & \text{if } \lambda = 1. \end{cases}$$

is a g-admissible weight. This example is adapted to a domain with conical points (for instance, a polygonal domain) with set of vertices $\mathcal{V}$ by taking $f(x) = \rho(x) := \prod_{P \in \mathcal{V}} |x - P|$ and $\lambda = 1$. The extra weight f becomes then unnecessary (for $\lambda = 1$) and the weighted Sobolev spaces $\mathcal{K}^{\ell}_{(\rho)}(M) := \{u \mid \rho^{|\alpha|}\partial^{\alpha} u \in L^2(M),\ (\forall)\, |\alpha| \le \ell\}$ are the spaces introduced by Kondratiev [17].

We shall *assume from now that* (M, g, ρ) *is an Amann triple (see Remark 3) and that* $f \colon M \to (0,\infty)$ *be a second g-admissible weight.* In particular, ρ is a bounded g-admissible weight. We have seen in Equation (2) how the Sobolev spaces change with respect to conformal changes of metric. Recall that $g_0 = \rho^2 g$. For differential operators, a simple calculation based on $L^\infty(M; E \otimes TM^{\otimes p} \otimes TM^{*\otimes q}, g) = \rho^{p-q} L^\infty(M; E \otimes TM^{\otimes p} \otimes TM^{*\otimes q}, g_0)$ and the fact that ρ is bounded gives:

Lemma 5. *Let P be an order k differential operator on M and $P_1 := \rho^k P$. We have that P satisfies the strong Legendre condition with respect to the metric g_0 if, and only if, P_1 satisfies the strong Legendre condition with respect to the metric $g = \rho^{-2} g_0$. If P has coefficients in $W^{\infty,\infty}(g_0)$, then P_1 has coefficients in $W^{\infty,\infty}(g)$.*

A similar result is valid for the boundary differential operators appearing as boundary conditions.

Regularity and well-posedness. Let P be a second order differential operator. We assume from now on that we are given a partition of the boundary $\partial M = \partial_0 M \sqcup \partial_1 M$ into two *disjoint, open subsets,* as in [5], and order i differential boundary conditions B_i on $\partial_i M$. See, for example, [7, 20] for general results on boundary value problems on smooth domains, [11, 18, 23, 22] for the case of non-smooth domains, and [4, 14] for more general boundary conditions involving projections. We assume that $\rho^2 P$, ρB_1, and B_0 *have coefficients in* $W^{\infty,\infty}(g)$. The typical assumption is that P, B_1, and B_0 have coefficients in $W^{\infty,\infty}(g_0)$, which means that they "stabilize" towards the singular points, as in [18], and this is a necessary condition for the existence of singular function expansions. In view of Lemma 5, our assumptions are thus weaker, but singular functions expansions are no longer available in general in our setting. Our more general setting may be needed in applications to nonlinear PDEs and uncertainty quantification. Also, recall from [14] the uniform Shapiro-Lopatinski regularity conditions and that they are invariant with respect to conformal changes of metric. Combining this property with Equation (2) and with Lemma 5, we get:

Theorem 6. *Let P be a g_0-uniformly elliptic second order differential operator acting on sections of $E \to M$ and $B = (B_0, B_1)$ be a boundary differential operator. We assume that P and B satisfy the g_0-uniform Shapiro-Lopatinski regularity conditions. Then, for any $\ell \in \mathbb{N}$, there exists $C > 0$ such that, for all $u \in f\mathcal{K}^1_{(\rho)}(M, g_0; E)$*

$$\begin{aligned}\|u\|_{f\mathcal{K}^{\ell+1}_{(\rho)}(M,g_0;E)} \leq C\big(&\|Pu\|_{f\rho^{-2}\mathcal{K}^{\ell-1}_{(\rho)}(M,g_0;E)} + \|u\|_{f\mathcal{K}^1_{(\rho)}(M,g_0;E)} \\ &+ \|B_0u\|_{f\rho^{-\frac{1}{2}}\mathcal{K}^{\ell+\frac{1}{2}}_{(\rho)}(\partial_0 M,g_0;E)} + \|B_1u\|_{f\rho^{-\frac{3}{2}}\mathcal{K}^{\ell-\frac{1}{2}}_{(\rho)}(\partial_1 M,g_0;E)}\big).\end{aligned}$$

In particular, we can take P to be a uniformly strongly elliptic scalar operator (such as the Laplacian $P = \Delta_{g_0}$), $B_0u = u|_{\partial_D M}$ (the restriction) and $B_1u = \partial^a_\nu u$.

Let (M, g_0) be a Riemannian manifold with boundary. We now turn to the well-posedness on (M, g_0). Let $h\colon M \to (0, \infty)$, and $A \subset \partial M$ be a measurable subset. We shall say that (M, A, E, g_0, h) satisfies the *Hardy-Poincaré inequality* if there exists a constant $C > 0$ such that, for any $u \in H^1_{\text{loc}}(M, g_0; E)$, $u = 0$ in $L^2(A)$, we have $\int_M |du|^2_{g_0} \,\text{dvol}_{g_0} \geq C \int_M h^{-2}u^2 \,\text{dvol}_{g_0}$. The Hardy-Poincaré inequality implies coercive estimates, and hence well-posedness also for the associated parabolic and hyperbolic equations, as in [20]. The Hardy-Poincaré inequality is related to the Poincaré inequality, and hence to the concept of "finite width." If $A \subset \partial M$, recall that (M, A) is said to have *finite width* if $\text{dist}(x, A)$ is uniformly bounded on M [5] (the distance between two disjoint connected components of M is $+\infty$). Typically in our results, the set A will be an open and closed subset of ∂M.

Example 7. Again, a typical application is when g_0 is the euclidean metric on $\mathbb{R}^m$, r is the distance to the origin, and $\lambda > 0$, as in Example 4. However, in this case, we let $\rho = r^\lambda$ *only* for $r < 1/2$, but set $\rho = r$ for $r > 1$ and $M \subset \mathbb{R}^m$ is a closed, infinite cone with base a smooth domain of the unit sphere and with vertex at the origin. Then again, (M, g) has bounded geometry if, and only if, $\lambda \geq 1$. Also, $(M, \partial M, g)$ has finite width if, and only if $\lambda = 1$. Finally, $(M, \partial M, g_0, \rho)$ satisfies the Hardy-Poincaré inequality (for ρ) if, and only if, $\lambda \leq 1$.

Recall that we have assumed (M, g) to be a Riemannian manifold with boundary and bounded geometry, $\rho, f\colon M \to (0, \infty)$ to be g-admissible weights, and $g_0 := \rho^2 g$. We define P_a by $(P_a u, v)_{g_0} := \int_M a(\nabla u, \nabla v) d\,\text{vol}_{g_0}$, with a sesquilinear form a satisfying the *strong Legendre condition with respect to* g_0. Let ∂^a_ν be the conormal derivative associated to P, see [14]. Combining Theorem 6 with the Lax-Milgram Lemma and with the fact that the Dirichlet and Neumann boundary conditions satify the uniform Shapiro-Lopatinski regularity conditions [5, 14], we obtain:

Theorem 8. *We assume that $(M, \partial_0 M, E, g_0, \rho)$ satisfies the Hardy-Poincaré inequality. Let $P = P_a$ satisfy the strong Legendre condition with all $\nabla^j a$ bounded.*

Then there exists $\eta_{a,f} > 0$ such that, for $|s| < \eta_{a,f}$ and $\ell \geq 1$, we have an isomorphism

$$P_a \colon \rho f^s \mathcal{K}^{\ell+1}_{(\rho)}(M, g_0; E) \cap \{u|_{\partial_0 M} = 0, \partial^a_\nu u|_{\partial_1 M} = 0\} \to \rho^{-1} f^s \mathcal{K}^{\ell-1}_{(\rho)}(M, g_0; E).$$

In particular, we can take $P = \Delta_{g_0}$, the Laplacian associated to g_0. For $\ell = 0$ the result remains true, once one reformulates it in a variational (i.e. weak) sense.

Examples. We include some basic examples.

Two dimensional domains. We consider a (disjoint) partition of the boundary $\partial M = \partial_0 M \sqcup \partial_1 M$ as above (so $\partial_0 M$ and $\partial_1 M$ are open and closed). Recall that P_a is a second order differential operators on $E \to M$ with coefficients in $W^{\infty,\infty}(g)$ and satisfying the strong Legendre condition with respect to g_0. For dimension two domains M, the Poincaré inequality for (M, A, g) is equivalent to the Poincaré inequality for (M, A, g_0) (same proof as the conformal invariance of the Laplacian in two dimensions). The Poincaré inequality of [5] then gives:

Theorem 9. *Assume that $(M, \partial_0 M, g)$ has finite width and $m := \dim(M) = 2$. Let $P = P_a$ satisfy the strong Legendre condition with all $\nabla^j a$ bounded. Then there exists $\eta_{a,f} > 0$ such that, for $|s| < \eta_{a,f}$ and $\ell \in \mathbb{Z}_+$, we have an isomorphism*

$$P_a \colon \rho f^s \mathcal{K}^{\ell+1}_{(\rho)}(M, g_0; E) \cap \{u|_{\partial_0 M} = 0, \partial^a_\nu u|_{\partial_1 M} = 0\} \to \rho^{-1} f^s \mathcal{K}^{\ell-1}_{(\rho)}(M, g_0; E).$$

In particular, we can take $P_a = \Delta_{g_0}$.

Canonical cuspidal and wedge domains. We continue with some concrete examples. The simplest examples in higher dimensions are those of "model cuspidal and wedge domains." We follow the presentation in [1]. Let $1 < \alpha < \infty$ and $B \subset \mathbb{R}^{m-1}$ a compact submanifold, possibly with boundary, and

$$K^m_\alpha(B) := \{(r, r^\alpha y) \in \mathbb{R}^m \mid 0 < r < 1, y \in B\}, \tag{3}$$

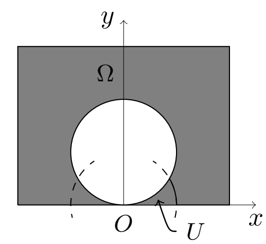
which will be called a *model canonical cusp of order α*. For $\alpha = 1$, we take B a subset of the unit sphere. A domain with *canonical cuspidal singularities* is a bounded domain $\Omega \subset \widehat{M}$ in a Riemannian manifold $(\widehat{M}, g_0)$ such that, around each singular point P of the boundary, it is locally diffeomorphic to $K^m_{\alpha_P}(B_P)$ via a diffeomorphism defined in a neighborhood of the ambient manifold. Let $\mathcal{V}$ be the set of singular points of the boundary, then $\mathcal{V}$ is finite and we let $M := \overline{\Omega} \smallsetminus \mathcal{V}$. If $\alpha_P = 1$ for all $P \in \mathcal{V}$, we obtain a domain with *conical points*. If we replace $K^m_\alpha(B)$ with $K^{m-k}_\alpha(B) \times [0,1]^k$, $k \geq 0$, we obtain domains with *canonical wedge singularities,* in which case, of course, the set $\mathcal{V}$ of singular points of $\partial\Omega$ will no longer be finite.

Let us fix $\lambda_P \geq 1$ for each singular point $P \in \mathcal{V}$. The weight functions ρ and f are then chosen, around each $P \in \mathcal{V}$ as in Example 4 for $\lambda = \lambda_P$. Let $g := \rho^{-2} g_0$, as before. If $\lambda_P \geq \alpha_P$ for all P, then (M, g) has bounded geometry (proved in [2] if $\lambda_P = \alpha_P$ for all $P \in \mathcal{V}$) and consequently, we have regularity in the weighted spaces for the mixed Dirichlet-Neumann problem for operators satisfying the strong Legendre condition. If $\lambda_P \leq \alpha_P$ and $\partial_0 M$ intersects each V_P, then $(M, \partial_0 M, g)$ satisfies the Hardy-Poincaré inequality. This follows from the usual Poincaré inequality on each $\{r\} \times r^\alpha B$ by also rescaling in r. We work in generalized spherical coordinates $(r, y) \in (0, \infty) \times S^{m-1}$, so $dx = r^{m-1} dr dy$. This gives $r^{-2\alpha} \int_{r^\alpha B} |u(r,y)|^2 dy \leq C \int_{r^\alpha B} |\nabla_y u(r,y)|^2 dy$ the Hardy-Poincaré inequality $\int_{K^m_\alpha(B)} r^{-2\alpha} |u(r,y)|^2 dx \leq C \int_{K^m_\alpha(B)} |\nabla u(r,y)|^2 dx$ for $u = 0$ on $\partial_1 B$. Let us fix

$\lambda_P = \alpha_P$. By considering the Amann triple $(M, g := \rho^{-2}g, \rho)$, we obtain that our domain with canonical wedge singularities satisfies the conclusion (isomorphism) of Theorem 9. For canonical cuspidal domains and constant coefficient operators, this theorem was first proved in [18]. See also [9, 10, 12, 16, 21].

Other examples. Certain simple examples are not "canonical."

Example 10. Let $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x, y > 0, (y-1)^2 + x^2 > 1\} \cap [-2,2] \times [0,2]$ (see the picture). The corners $\{A_i\}_{i=1}^4$ of Ω are conical points and are treated as above.



Close to O we have two cuspidal open sets (U and its mirror image), similar to the ones treated in the previous subsection, but *not canonical*. We have $\lambda = 2$ for these open sets. (We double O, in a certain sense.) Let r be the distance to O. Close to O, we then choose $\rho \sim r^2$, and $f \sim e^{-r^{-1}}$. More precisely, $\rho(x) = r^2 \prod_{i=1}^4 |x - A_i|$ and $f = e^{-r^{-1}} \prod_{i=1}^4 |x - A_i|$. Note that we could have also chosen $\rho \sim x^2$ near O. Let $g := \rho^{-2}g_E$, where g_E is the standard (flat) Euclidean metric. We then have that (M, g), $M := \overline{\Omega} \smallsetminus \{O, A_i, i = 1, 4\}$, is a manifold with boundary and bounded geometry. Assume that $\partial_0 M$ touches each singular point (where O is considered as a double point as above). We can then prove that $(M, \partial_0 M, g_E)$ satisfies the Hardy-Poincaré inequality as in the previous example, and hence Theorem 9 applies. See also [16, 21].

Example 11. Let $f_0, f_1 \colon \mathbb{R} \to (0, 2\pi)$ satisfy $\|f_i^{(k)}\|_\infty < \infty$, $k \geq 0$, and $f_1 - f_0 \geq \epsilon > 0$. Let $\Omega_{f_0,f_1} := \{(r\cos\theta, r\sin\theta) \mid f_0(\log r) < \theta < f_1(\log r)\}$ and Ω be a bounded domain, smooth away from a finite number of points, at which it coincides, up to a diffeomorphism, with a neighborhood of 0 in a set of the form Ω_{f_0,f_1}. Then Ω is a domain with *oscillating conical singularities,* similar to the ones studied by [24]. Theorem 9 holds for this domain with $\rho = f = r = (x^2 + y^2)^{1/2}$. In general, a domain with *oscillating* conical points *is not* a domain with conical points.

Finally, in [8], it was proved that the assumptions and the conclusions of Theorem 8 are fulfilled by any polyhedral domain $\Omega \subset \mathbb{R}^m$ (defined as a suitable stratified space) with the Euclidean metric g_0 for a suitable g-admissible weight $\rho \sim$ the distance to the singular points of the boundary ($g = \rho^{-2}g_0$).

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