A COMPARISON OF THE GEORGESCU AND VASY SPACES ASSOCIATED TO THE N-BODY PROBLEMS AND APPLICATIONS

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ABSTRACT. We provide new insight into the analysis of N-body problems by studying a compactification M_N of $\mathbb{R}^{3N}$ that is compatible with the analytic properties of the N-body Hamiltonian H_N. We show that our compactification coincides with a compactification introduced by Vasy using blow-ups in order to study the scattering theory of N-body Hamiltonians and with a compactification introduced by Georgescu using C^*-algebras. In particular, the compactifications introduced by Georgescu and by Vasy coincide (up to a homeomorphism that is the identity on $\mathbb{R}^{3N}$). Our result has applications to the spectral theory of N-body problems and to some related approximation properties. For instance, results about the essential spectrum, the resolvents, and the scattering matrices of H_N (when they exist) may be related to the behavior near $M_N \smallsetminus \mathbb{R}^{3N}$ (i.e. "at infinity") of their distribution kernels, which can be efficiently studied using our methods. The compactification M_N is compatible with the action of the permutation group S_N, which allows to implement bosonic and fermionic (anti-)symmetry relations. We also indicate how our results lead to a regularity result for the eigenfunctions of H_N.

CONTENTS

1. INTRODUCTION

1.1. **A general introduction and motivation for our work.** The quantum behavior of an atomic system is often investigated via its associated Hamiltonian. A good model for N non-relativistic particles interacting with each other by Coulomb type forces is given by the Hamiltonian

$$(1) \qquad (H_N u)(x) := \Big(-\sum_{j=1}^{N} \frac{1}{2m_j} \Delta_{x_j} + \sum_{1 \le j < k \le N} \frac{b_{jk}}{|x_j - x_k|} \Big) u(x) \,,$$

where $x_j \in \mathbb{R}^3$ describes the position of the j-th particle, $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{3N}$, the operator Δ_{x_j} is the Laplacian with respect to x_j, $m_j \in \mathbb{R}_+$, and $b_{jk} \in \mathbb{R}$. See, for instance, [6, 23]. As usual, by moving to the center of mass coordinates and effective operators, to an atom with $N - 1$ electrons (corresponding to N "bodies") will correspond an operator H_{N-1}^{eff} acting on functions defined on $\mathbb{R}^{3(N-1)}$.

The way the mathematical properties of Hamiltonians are reflected in the properties of the physical system was explained in many works, including [6, 23, 26, 41, 54, 68]. In particular, the mathematical study of the operator H_N (and of its simplified version H_{N-1}^{eff}) is a very vast domain of study in quantum mechanics and in mathematics. We will not be able to do justice to all the people who have contributed to the field, but let us nevertheless mention some works that are among the closest to the methods of this paper, namely the monographs of Amrein, Boutet de Monvel, and Georgescu [6], Dereziński and Gérard [25] and Teschl [69], as well as the research papers [7, 20, 23, 24, 31, 32, 34, 58]. More specific references even closer related to our work can be found below.

The mathematical study of H_N and H_{N-1}^{eff} is quite challenging, especially for $N > 2$. The simplest case is that of hydrogen type atoms, which corresponds to $N = 2$ and $m_1 \gg m_2$. Then

$$(2) \qquad H_1^{\mathrm{eff}} u(x) := \Big(-\frac{1}{2\mu} \Delta + \frac{b}{|x|} \Big) u(x) \,, \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \,.$$

In order to understand the mathematical properties of this operator, one usually writes the Hamiltonian H_1^{eff} in spherical coordinates

$$(r, y) \in (0, \infty) \times \mathbb{S}^2, \quad r = |x|, \quad x = ry,$$

where $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$, as usual. The use of spherical coordinates has led, for instance, to the determination of the spectrum of H_1^{eff} and to explicit formulas for its eigenfunctions (see [25, 28, 69, 73] for details and historical references), which is the basis for the orbital model in (quantum) chemistry. For $N > 2$ explicit calculations seem to be impossible. Nevertheless, one can still try to find "more convenient coordinates" in which to perform our calculations than the usual, euclidean coordinates. In this vein, one of the main results of this paper is to provide convenient coordinates that generalize the polar coordinates and which are helpful to study both particle interactions at infinity and the regularity of eigenfunctions for $N > 2$ particles.

More precisely, our "more convenient coordinates" (in the case of N-particles and the effective Hamiltonian) patch together to yield a compact (smooth) manifold with corners M_{N-1}, whose interior $M_{N-1} \setminus \partial M_{N-1}$ is $\mathbb{R}^{3(N-1)}$. Thus M_{N-1} is a *compactification* of $\mathbb{R}^{3(N-1)}$ in the usual mathematical sense. In turn, the construction of such a compactification M_{N-1} may possibly yield more convenient coordinate systems via its natural coordinate charts. For the hydrogen atom, $N = 2$ and M_1 is the *radial compactification* $M_1 := \overline{\mathbb{R}^3} = \mathbb{R}^3 \cup \mathbb{S}_{\mathbb{R}^3}$ (see Subsection 5.1 for the definition of the radial compactification). In the earlier literature, two compactifications of $\mathbb{R}^{3(N-1)}$ for $N > 1$ have played an especially important role in the study of the N-body problem, a role which will be explained now.

To do that, it will be convenient to place ourselves in a slightly more general setting in which we consider a finite semilattice $\mathcal{F}$ of linear subspaces a finite dimensional, real vector space X. In this setting, a first such compactification is Georgescu's compactification, which was obtained as the primitive ideal spectrum $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ of a certain commutative C^*-algebra [14] $\mathcal{E}_{\mathcal{F}}(X)$ [32, 34, 35]. Another compactification, $[X : \mathbb{S}_{\mathcal{F}}]$ was constructed by Vasy using iterated blow-ups [70]. See also [48]. The constructions of these compactifications and the notation will be explained shortly in Subsection 1.3. One of the *main results* of this paper is to show that both Georgescu's compactification and Vasy's compactification are naturally homeomorphic with the one introduced in [61] (see Equation (7)). This common compactification will be called the "Georgescu-Vasy" space and will be denoted X_{GV}. In case $\mathcal{F}$ is the semilattice generated by the collision planes of the N-body problem for the effective Hamiltonian, then the Georgescu-Vasy compactification of $\mathbb{R}^{3(N-1)}$ will be the space M_{N-1} that provides the more convenient coordinates were looking for. The identification of Georgescu's and Vasy's compactifications for the N-body problem with the same space M_{N-1} will allow us to obtain further properties for the corresponding Hamiltonian since, as we will explain below, each of the constructions of M_{N-1} mentioned above has its own advantages. Of course, the role of the compactifications in the work of these authors – as well as in ours – is to use the the properties of the space M_{N-1} to obtain a better insight into the properties of the Hamiltonian H_{N-1}^{eff} (or of related Hamiltonians). In this spirit, we present some applications of our results in Section 6. These applications are also summarized at the end of this introduction.

1.2. **Our setting and our construction of the compactification** M_N**.** Our results will be mostly for operators that are somewhat different from the N-body Hamiltonians H_N or H_{N-1}^{eff} of Equations (1) and (2). The operators that we study are, in fact, in many regards, more general than the N-body Hamiltonians and, in any case, they retain most of the main

features of the operators H_N and H_{N-1}^{eff} that are relevant to this paper. To describe the class of operators that we will study, let us thus first explain the following two customary modifications of the operators H_N, following, for instance, [6, 23, 32] and the references therein.

The first modification is to "smooth out" the singularity in the potential. To see why this is reasonable, let us mention that, from the point of view of Partial Differential Equations, there are two main issues that distinguish H_N and H_{N-1}^{eff} from the customary differential operators studied in the introductory courses, namely:

- the behavior at infinity of the potential and
- the singularities in the potential.

Somewhat counterintuitively (from a pure mathematical point of view) is that the singularities of the potential are less important than the behavior at infinity, at least in what the spectral theory is concerned. In fact, it is known that many results concerning the essential spectrum of operators with potentials with Coulomb singularities can be obtained from the results for the analogous operators, but with smooth potentials (see, for example, [35] and the references therein; incidentally, in addition to the Hardy inequality, the argument there involves norm closures and elementary C^*-algebra results). Thus, except in the very last subsection, in this paper, we will "smooth out" the singularities and hence look instead at a class of operators containing operators of the form

$$(3) \qquad H_N' \; := \; D + \sum_{1 \leq j \leq N} v_j(x_j) + \sum_{1 \leq j < k \leq N} v_{jk}(x_j - x_k) \,,$$

where D is a strongly elliptic differential operator with constant coefficients, and v_j and v_{jk} are *smooth functions* on $\mathbb{R}^3$ with uniform radial limits at infinity. (We suppressed from the notation the function u on which H_N' acts. Also, our results remain valid for certain operators D with suitable, non-constant coefficients.) In this paper, by a function with "uniform radial limits at infinity" we mean a function that extends to a smooth function on the radial compactification of that space, $\mathbb{R}^3$ in this case. The choice of functions with uniform radial limits at infinity is motivated by the choice of "more convenient coordinates" in the case of the hydrogen type atom and also because it leads to a less singular compactification of $\mathbb{R}^{3N}$. We note, however, that in the last subsection, the singularities of the potential will play a central role in our regularity estimates of Equation (10).

Our second modification of the operators H_N (already alluded to above) will be to extend our setting from $\mathbb{R}^{3N}$ to an arbitrary real, finite dimensional vector space X and to allow for more general collision planes. (More general than the collision planes $\{x_j - x_k = 0\}$ for H_N). More precisely, we will allow our collision planes to belong to a suitable finite set $\mathcal{F}$ of linear subspaces of X, as above and as in [6, 23, 25, 34], for instance. As in those works, which serve as a motivation for our approach, it will be convenient to assume that $\mathcal{F}$ is stable under intersection. Recall that a family $\mathcal{S}$ of subsets of M is a *semilattice* (with respect to the inclusion) if, for all $P_1, P_2 \in \mathcal{S}$, we have $P_1 \cap P_2 \in \mathcal{S}$. It will also be convenient – and that will not decrease the generality – to consider semilattices $\mathcal{F}$ with $\{0\} \in \mathcal{F}$. Our operator H_N' will then be replaced with a more general operator of the form

$$(4) \qquad H \; := \; D + \sum_{Y \in \mathcal{F}} v_Y \,,$$

where v_Y is a smooth function on X/Y with uniform radial limits at infinity (more precisely $v_Y \in \mathcal{C}(\overline{X/Y})$, where X/Y is the quotient vector space and $\overline{X/Y}$ is its radial compactification). This completes our sequence of modifications of H_N and provides us

with the concepts needed to introduce our definition of compactification space X_{GV} (the Georgescu-Vasy space) as follows.

Let $\mathcal{F}$ be a finite semilattice of linear subspaces of X with $\{0\} \in \mathcal{F}$, as above, and let

$$(5) \qquad \delta_{\mathcal{F}} : X \to \prod_{Y \in \mathcal{F}} \overline{X/Y}$$

be the diagonal map obtained from all the projections $X \to X/Y$. We define the *Georgescu-Vasy space* X_{GV} as the closure

$$(6) \qquad X_{\mathrm{GV}} := \overline{\delta_{\mathcal{F}}(X)}$$

of $\delta_{\mathcal{F}}(X)$ in $\prod_{Y \in \mathcal{F}} \overline{X/Y}$. Since each $\overline{X/Y}$ is compact, X_{GV} is also compact. Note that, our assumption that $\{0\} \in \mathcal{F}$ implies, in particular, that the factor $\overline{X/Y}$ corresponding to $Y = \{0\}$ is $\overline{X/\{0\}} = \overline{X}$, which contains X as a dense, open subset, and hence the map $\delta_{\mathcal{F}}$ is injective on X. If $\mathcal{F}$ is the semilattice corresponding to the N-body problem or the effective N-body problem, then X_{GV} will yield the desired space M_k (for suitable k).

1.3. **Georgescu's and Vasy's compactifications.** Let us recall first the definition of Georgescu's compactification [6, 32, 34] in the form considered in [35]. To this end, let us consider the norm closed algebra (C^*-algebra)

$$(7) \qquad \mathcal{E}_{\mathcal{F}}(X) := \langle \mathcal{C}(\overline{X/Y}) \rangle$$

generated by all the spaces $\mathcal{C}(\overline{X/Y})$ in $L^\infty(X)$, with $Y \in \mathcal{F}$. (Here $\mathcal{C}(Z)$ denotes the space of continuous functions $Z \to \mathbb{C}$, as usual.) The spectrum $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ of this algebra (the set of its characters) is a compact space containing naturally X. *Georgescu's compactification* is $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$. It was proved in [61] using elementary C^*-algebra arguments that $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ is naturally homeomorphic to the spaces $X_{\mathrm{GV}} := \overline{\delta_{\mathcal{F}}(X)}$ considered above. A variant of the algebra $\mathcal{E}_{\mathcal{F}}(X)$ defined above is obtained by considering the one-point compactifications $(X/Y)^+$ [6, 15, 32], see Remark 5.12. Also, in [62], the definition of the algebra $\mathcal{E}_{\mathcal{F}}(X)$ was generalized so that the only requirement on $\mathcal{F}$ is $\{0\} \in \mathcal{F}$. In particular, the family $\mathcal{F}$ can be infinite and not necessarily stable by intersection (but the sum defining the potential must be convergent).

Vasy's compactification is obtained using blow-ups of manifolds with corners. Let M be a manifold with corners (we recall the definition of manifolds with corners and related constructions in Section 2). Recall that a *p-submanifold* $P \subset M$ is a submanifold of M that has a tubular neighborhood: $P \subset U_P \subset M$ that is locally of a product form (see Definition 2.15 for details). If P is a closed p-submanifold – where "closed" means closed as a subset – then the *blow-up* $[M : P]$ *of M with respect to P* is defined by replacing P with the set $\mathbb{S}(N_+^M P)$ of *interior* directions in the normal bundle $N^M P$ of P in M (see [1, 43, 56, 66], or Definiton 3.1). As in those papers, for a suitable k-tuple $(P_j)_{j=1}^k = (P_1, P_2, \ldots, P_k)$, we can define the iterated blow-up $[M : (P_j)_{j=1}^k]$ by blowing up M first with respect to P_1, then with respect to (the lift of) P_2 to $[M : P_1]$, and then continuing in this way (see Definition 4.1). We apply these results to the study of the N-body problem in the following way. Let $\overline{X}$ denote be the spherical compactification of a finite-dimensional vector space X, with boundary at infinity the sphere $\mathbb{S}_X := \overline{X} \smallsetminus X$ (see Subsection 5.1 for the detailed definitions). To our finite semilattice $\mathcal{F}$ of linear subspaces of X containing the zero subspace we associate the semilattice

$$(8) \qquad \mathbb{S}_{\mathcal{F}} := \{\mathbb{S}_Y = \mathbb{S}_X \cap \overline{Y} \mid Y \in \mathcal{F}\}.$$

(Note that our assumption that $\{0\} \in \mathcal{F}$ implies that $\emptyset = \mathbb{S}_{\{0\}} \in \mathbb{S}_{\mathcal{F}}$. Our approach, in fact, works also for semilattices that do not contain the zero subspace, but, in any case, we can reduce to this case by including the zero subspace in $\mathcal{F}$.) We arrange the elements of $\mathbb{S}_{\mathcal{F}}$ according an "admissible order": $\mathbb{S}_{\mathcal{F}} = (P_0 = \emptyset, P_1, P_2, \ldots)$ (roughly, in the ascending inclusion order, see Definition 4.12). Then *Vasy's compactification* is obtained by iteratively blowing up $\overline{X}$ with respect to $\mathbb{S}_{\mathcal{F}}$ as explained earlier to obtain $[\overline{X} : \mathbb{S}_{\mathcal{F}}]$ [71, 72]. (This construction is discussed in detail in the main body of the paper, Definition 4.1.) See also Kottke's paper [48], where this compactification was also recently studied.

Yet a fourth compactification of X, still diffeomorphic to X_{GV}, is the *graph blow-up* $\{\overline{X} : \mathbb{S}_{\mathcal{F}}\}$. The *graph blow-up* $\{M : \mathcal{P}\}$ of M with respect to the family $\mathcal{P}$ is the closure $\{M : \mathcal{P}\} := \overline{\delta(M \smallsetminus \bigcup_{P \in \mathcal{P}} P)} \subset \prod_{P \in \mathcal{P}}[M : P]$ where δ is the diagonal embedding (see Definition 4.17). One of the main results of this paper, Theorem 4.19, establishes a diffeomorphism $[M : (P_j)_{j=1}^k] \simeq \{M : (P_j)_{j=1}^k\}$ between the iterated blow-up and the graph blow-up introduced in Definition 4.17, provided that $(P_j)_{j=1}^k$ is an admissible ordered clean semilattice. This compactification arises as an intermediate step in our proof of the equivalence (homeomorphism) of Georgescu's construction (the space $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$) and Vasy's construction (the space $[\overline{X} : \mathbb{S}_{\mathcal{F}}]$) that relies, in order, on the following sequence of homeomorphisms

$$(9) \qquad [\overline{X} : \mathbb{S}_{\mathcal{F}}] \simeq \{\overline{X} : \mathbb{S}_{\mathcal{F}}\} \simeq X_{\mathrm{GV}} := \overline{\delta_{\mathcal{F}}(X)} \simeq \mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X)) \,.$$

The smooth structures on the last three spaces come from the first one. We stress that, while the last homeomorphism (already proved in [61]) has an easy proof, the other two, proved in this paper, are rather difficult results.

1.4. Contents of the paper and applications.
Let us now present the contents of the paper, including some applications of our results (mainly of the homeomorphisms of Equation (9)). Section 2 contains background material on manifolds with corners. In particular, we devote quite a bit of effort to introduce and compare several classes of submanifolds of manifolds with corners. Section 3 recalls the definition of the blow-up of a manifold with corners with respect to a closed p-submanifold and establishes a few properties of this blow-up. In Section 4 we study the iterated and the graph blow-ups. In particular, we prove that, for clean semilattices of closed p-submanifolds, the graph blow-up can also be obtained as an iterated blow-up, which is one of the main technical results of this paper. In Section 5, we use the identification of the graph blow-up with the iterated blow-up to show that Georgescu's compactification $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ and Vasy's compactification $[\overline{X} : \mathbb{S}_{\mathcal{F}}]$, are homeomorphic to the space X_{GV} introduced above (see Equation (9)).

An important application of our results is the existence of various smooth group actions on blow-ups, in general, and on X_{GV}, in particular. For instance, we obtain an action of X on X_{GV} by translations. Moreover, when $\mathcal{F}$ corresponds to the N-body problem, we also obtain and action of the symmetric group S_N and of the orthogonal group $\mathrm{GL}(3, \mathbb{R})$ on $M_N = X_{\mathrm{GV}}$, see Remark 6.2. In addition to group-action applications, in the last section, we outline *four other applications* of our results, which we resume next.

(A) The first application is on the relation between the action of the symmetric group S_N on M_N and Pauli exclusion principle. Finding a good compactification of $\mathbb{R}^{3N}$ that behaves well with respect to the action of S_N is a problem posed by Melrose and Singer from [59], which was solved in [48] using differential geometry and in [62] using C^*-algebras, see Subsection 6.1.

(B) The second application is to investigate the relation between Vasy's pseudodifferential calculus and Georgescu's algebras. It combines the results of [4] with the existence of the smooth structure on X_{GV} provided by our results and with the smooth action of X on X_{GV} to define a pseudo-differential calculus $\Psi_c(X_{\mathrm{GV}})$ consisting of properly supported operators. This calculus is clearly smaller than the Vasy calculus, and a preliminary discussion of the relation of the two calculi is contained in Subsection 6.2. We thus also define a completion $\Psi_{NB}(X_{\mathrm{GV}})$ of $\Psi_c(X_{\mathrm{GV}})$ that can be proved to be spectrally invariant (*i.e.,* it is stable for holomorphic functional calculus) and thus leads to a description of the distribution kernels of the resolvents of H_N, Proposition 6.3. The more precise relation between Vasy's calculus and ours is, certainly, worth further exploring, see Subsection 6.2.

(C) Our third application is to establish some connections between our results and the HVZ theorem, see Subsection 6.3

(D) Finally, the last application is a regularity result for bound states for Hamiltonians with *inverse square singularities*. For instance, let $u \in L^2(X \smallsetminus \cup \mathcal{F})$ be an eigenfunction of H_N, namely $H_N u = \lambda u$ on $X \smallsetminus \cup \mathcal{F}$, $\lambda \in \mathbb{R}$, and let $\rho(x) := \min\big\{\operatorname{dist}(x, \cup \mathcal{F}), 1\big\}$, where $\operatorname{dist}(x, \cup \mathcal{F})$ is the distance in the usual euclidean metric from x to $\cup \mathcal{F}$. Then, for any multi-indices α, we have

$$(10) \qquad \rho^{|\alpha|} \partial^\alpha u \in L^2(\mathbb{R}^{3N}).$$

The same result holds for H_N^{eff} in place of H_N and for many other operators. See Theorem 6.4 in Subsection 6.4 for a more general statement. Full details can be found in [5].

The study of inverse-square potentials is relevant since they appear in relativistic physics. See Subsection 6.4 for references and more on the physical motivation for inverse square potentials.

Two appendices include some related topological results on proper maps and on submanifolds of manifolds with corners. The reader can thus see that this paper relies essentially on geometry, necessarily so since Vasy's construction is geometric.

2. MANIFOLDS WITH CORNERS AND THEIR SUBMANIFOLDS

We begin with some background material, mostly about manifolds with corners. This section contains few new results, but the presentation is original.

2.1. **Manifolds with corners.** We now introduce manifolds with corners and their smooth structure. We also set up some important notation to be used throughout the paper. The terminology used for manifold with corners is not uniform. Nevertheless, good overviews of the concept of a manifold with corners can be found in [43, 47, 49, 57, 66], to which we refer for the concepts not defined here and for further references. In this paper, we will mostly use the terminology introduced by Melrose and his coauthors, which predates most of the other ones.

2.1.1. *Notation and conventions.* For any finite-dimensional real vector space Z, let $\mathbb{S}_Z$ denote the *set of vector directions* in Z, that is, the set of (non-constant) open half-lines $\mathbb{R}_+ v$, with $0 \neq v \in Z$ and $\mathbb{R}_+ := (0, \infty)$. We will also use the standard notation $\mathbb{S}^{n-1} := \mathbb{S}_{\mathbb{R}^n}$, for simplicity. In particular, if Z is a euclidean (real) vector space, we identify $\mathbb{S}_Z$ with the unit sphere in Z. Informally, a manifold with corners is a topological space locally modeled on the spaces

$$(11) \qquad\qquad \mathbb{R}^n_k \; := \; [0, \infty)^k \times \mathbb{R}^{n-k}.$$

For $k, n \in \mathbb{N} = \{0, 1 \ldots\}$ with $k \leq n$, we let $\mathbb{S}^{n-1}_k \subset \mathbb{R}^n$ be

$$(12) \quad \mathbb{S}^{n-1}_k \; := \; \mathbb{S}^{n-1} \cap \mathbb{R}^n_k \; = \; \{\phi = (\phi_1, \ldots \phi_n) \,|\, \|\phi\| = 1 \text{ and } \phi_i \geq 0 \text{ for } 1 \leq i \leq k\},$$

where $\|.\|$ is the euclidean norm on $\mathbb{R}^n$.

Remark 2.1. Let us write 0_V for the neutral element of a vector space V, when we want to stress the space to which it belongs. We will often use maps between subsets of euclidean spaces. As a rule, we will try not to permute the coordinates and, moreover, our embeddings will be "first components" embeddings. More precisely, let $k' \leq k$ and $n'-k' \leq n-k$, we shall then use with priority the canonical *first components embedding,* namely given by:

$$(13) \qquad \begin{aligned} \mathbb{R}^{n'}_{k'} &\simeq [0, \infty)^{k'} \times \{0_{\mathbb{R}^{k-k'}}\} \times \mathbb{R}^{n'-k'} \times \{0_{\mathbb{R}^{n-n'}}\} \subseteq [0, \infty)^k \times \mathbb{R}^{n-k} \; = \; \mathbb{R}^n_k \\ (x', x'') &\mapsto (x', 0_{\mathbb{R}^{k-k'}}, x'', 0_{\mathbb{R}^{n-n'}}) \,. \end{aligned}$$

Occasionally, other embeddings (involving permutations of the coordinates) between these sets will also be considered, in which case they will be explained separately. For instance, we shall sometimes find it notationally convenient to use the *canonical permutation of coordinates* diffeomorphism

$$(14) \qquad \begin{aligned} \operatorname{can} : \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} &\simeq \mathbb{R}^{n+n'}_{k+k'} \\ (x', x'', y', y'') &\mapsto (x', y', x'', y'') \in [0, \infty)^{k+k'} \times \mathbb{R}^{n+n'-k-k'} \,, \end{aligned}$$

where $x' \in [0, \infty)^k$ and $y' \in [0, \infty)^{k'}$. (Compare with Equation (11).)

2.1.2. *Charts and atlases.* We shall use suitable charts to define the smooth structure on manifolds with corners. We proceed as in the case of smooth manifolds (without corners). We begin with the following standard definition.

Definition 2.2. Let $U \subset \mathbb{R}^n_k$ and $V \subset \mathbb{R}^m_l$ be two open subsets and $f = (f_1, \ldots, f_m) : U \to V$. We shall say that:

   (a) f is *smooth* on U if there exists an open neighborhood W of U in $\mathbb{R}^n$ such that f extends to a smooth function $\tilde{f} : W \to \mathbb{R}^m$.
   (b) f is a *diffeomorphism* between U and V if f is a bijection and both f and f^{-1} are smooth.

Definition 2.3. A *corner chart on M* (or simply, *"chart"* in what follows) is a couple (U, ϕ) with U an open subset of M and $\phi : U \to \Omega$ a homeomorphism onto an open subset Ω of $\mathbb{R}^n_k$. Let (U, ϕ) and (U', ϕ') be two corner charts with values in $\mathbb{R}^n_k$ and in $\mathbb{R}^{n'}_{k'}$, respectively. Let $V := U \cap U'$. We shall say that the corner charts (U, ϕ) and (U', ϕ') are *compatible* if

$$\phi' \circ \phi^{-1} : \phi(V) \to \phi'(V)$$

is a diffeomorphism (see Definition 2.2) between the open subsets $\phi(V) \subset \mathbb{R}^n_k$ and $\phi'(V) \subset \mathbb{R}^{n'}_{k'}$. (So $n = n'$ if $V \neq \emptyset$.)

Given a point $m \in M$ and a corner chart (U, ϕ) with $m \in U$, we can always find a corner chart (U', ϕ'), $\phi' : U' \to \mathbb{R}^{n'}_{k'}$, compatible with (U, ϕ) such that $\phi'(m) = 0$ and k' is minimal. The least such k is the *boundary depth* of m in M. See Subsection 2.2 below.

Definition 2.4. A *corner atlas* $\mathcal{A} = \{(U_a, \phi_a), a \in A\}$ on M is a family of compatible corner charts such that $M = \bigcup\limits_{a \in A} U_a$. Two corner atlases are called *equivalent* if their union is again a corner atlas. A *manifold with corners* is defined to be a paracompact Hausdorff space M with an equivalence class of corner atlases (on M). In the following we will drop the word "corner" before the words "chart" and "atlas." In the context of a manifold with corners, the terms "atlas" and "chart" will always mean "corner atlas" and, respectively, "corner chart."

We stress that we do not require the connected components of a manifold with corners to have the same dimension. Let us also remark that a manifold with corners in the above sense is called a "*t*-manifold" in [56, Section 1.6], where the "*t*" stands for "tied". If M is manifold with corners, then the union of all its atlases is again an atlas, the *maximal atlas* of M. An open subset of a manifold with corners is, again, in an obvious way, a manifold with corners. Many concepts extend from the case of manifolds without corners to that of manifolds with corners.

Definition 2.5. Let $f : M \to M'$ be a map between two manifolds with corners. We will say that f is *smooth* if, for any two charts (U, ϕ) of M and (U', ϕ') of M', the map $\phi' \circ f \circ \phi^{-1}$ is smooth on its domain of definition $\phi(f^{-1}(U'))$. If f is a bijection and both f and f^{-1} are smooth, we will say that f is a *diffeomorphism*.

Here are some examples of manifolds with corners that will be used in this paper.

Example 2.6. Using the notation from Subsection 2.1.1, we have the following:

  (i) Any open subset of $\mathbb{R}^n_k := [0, \infty)^k \times \mathbb{R}^{n-k}$ is a manifold with corners.
 (ii) The sphere orthant $\mathbb{S}^{n-1}_k := \mathbb{S}^{n-1} \cap \mathbb{R}^n_k$ of Equation (12) is a manifold with corners.
(iii) Any smooth manifold is a manifold with corners (even if it does not have a boundary or any true corners).

The following will be used to introduce p-submanifolds.

Definition 2.7. Let I be a subset of $\{1, \ldots, n\}$ and L_I be the subset of $\mathbb{R}^n_k$ defined by

$$(15) \qquad\qquad L_I := \{\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n_k \mid x_i = 0 \text{ if } i \in I \,\}.$$

The number $b := |I \cap \{1, \ldots, k\}|$ of elements of $I \cap \{1, \ldots, k\}$ will be called the *boundary depth* of L_I in $\mathbb{R}^n_k$. Similarly, $c := |I|$ is the codimension of L_I in $\mathbb{R}^n_k$ and $d := n - c$ is its dimension.

2.2. **The boundary and boundary faces of a manifold with corners.** We now fix some standard terminology to be used in what follows, extending the local definitions of Definition 2.7. In particular, we need the intrinsic definition of the boundary of a manifold with corners. Recall that above we defined the *boundary depth* (in M) $\mathrm{depth}_M(p)$ of a point $p \in M$ as the number of non-negative coordinate functions vanishing at p in any local coordinate chart at p. It is the least k such that there exists a chart near U with values in $\mathbb{R}^n_k$. Let $(M)_k$ be the set of points of M of boundary depth k. It is a smooth manifold (no corners). Its connected components are called the *open* boundary faces (or just the *open* faces) of codimension (or boundary depth) k of M. The set $(M)_0$ is the *interior* of M. A *boundary face* of boundary depth k is the closure of an open boundary face of

FIGURE 1. A teardrop domain: a compact simply-connected domain
in $\mathbb{R}^2$ with exactly one boundary face F_1 of boundary depth 1 and ex-
actly one corner p of boundary depth 2. The face F_1 is the image of an
immersion of $[0, 1]$ that is injective on the open interval $(0, 1)$, but which
maps both ends to the corner p.

boundary depth k. For every boundary face F of codimension k there is a manifold with
corners $\mathbb{F}$ of dimension $n - k$ and an immersion $\iota \colon \mathbb{F} \to M$ that is injective on $(\mathbb{F})_0$ and
$\iota(\mathbb{F}) = F$. However, in general ι is not injective on the boundary faces of $\mathbb{F}$; an example
is the teardrop domain of Figure 2.2. For this domain, M does not induce a structure of
a manifold with corners on the boundary face F. Nevertheless, we will not exclude from
our consideration boundary faces that are not manifolds with corners; in particular, the
boundary of the teardrop domain is a boundary face of that domain.

We will denote by $\mathcal{M}_k(M)$ the set of all *closed* boundary faces of codimension k. In
particular, the *boundary* ∂M of M, defined as the set of all points of boundary depth > 0,
is given by

$$(16) \qquad \partial M := \bigcup_{H \in \mathcal{M}_1(M)} H.$$

A boundary face of M of codimension one will be called a *hyperface*, as usual. Thus ∂M
is the union of the hyperfaces of M. If H is a hyperface of M and $0 \le x \in \mathcal{C}^\infty(M)$ is
a function such that $H = x^{-1}(0)$ and $dx \ne 0$ on H, then x is called a *boundary defining
function* of H. As the example of the teardrop domain shows, not all hypersurfaces have a
boundary defining function. However, each boundary face F of codimension k can *locally*
be represented as

$$F \cap U = \{x_1 = x_2 = \ldots = x_k = 0\},$$

where x_j are boundary defining functions of the hyperfaces of $F \cap U$ in U.

It is convenient to consider embeddings.

Remark 2.8. Every manifold with corners M is contained in a smooth manifold $\widetilde{M}$ such
that each component of M is contained in a component of $\widetilde{M}$ of the same dimension
[2, 43, 49, 56, 57, 66]. We note, however, that ∂M is intrinsically defined, Equation (16),
and, sometimes, it is *not* the *topological* boundary $\overline{M} \smallsetminus \overset{\circ}{M}$ of M, where the closure $\overline{M}$
and the interior $\overset{\circ}{M}$ are defined as subsets of $\widetilde{M}$. For instance, when $M := \{x \in \mathbb{R}^n \mid x_n \ge 0, \|x\| < 1\}$ and $\widetilde{M} = \mathbb{R}^n$, then $\partial M = \{x \in \mathbb{R}^n \mid x_n = 0, \|x\| < 1\}$, whereas the
topological boundary of M is $\partial M \cup \{x \in \mathbb{R}^n \mid x_n > 0, \|x\| = 1\}$, a bigger set. In fact, we
always have that ∂M is contained in the topological boundary of M in $\widetilde{M}$. Unlike ∂M, the
topological boundary of M in $\widetilde{M}$ depends on $\widetilde{M}$. Embeddings are convenient also because
one can define

$$TM := T\widetilde{M}|_M.$$

Up to a diffeomorphism, TM can be obtained by gluing open subsets of the tangent spaces
$T(\mathbb{R}^n_k) := \mathbb{R}^n_k \times \mathbb{R}^n$ using an atlas of M. We also let $T_x^+ M$ be the set of *inward-pointing*

tangent vectors of $T_x M$ (this includes the vectors tangent to the boundary). It can be defined as the set of equivalence classes of curves starting at x and completely contained in M. We finally let $T^+ M := \bigcup_{x \in M} T_x^+ M$ with its projection map to M. Note that $T^+ M$ is *not* a fiber bundle, but a fiberwise conical closed subset of the tangent space.

2.3. **Submanifolds of manifolds with corners.** We now discuss the two notions of submanifolds of a manifold with corners – "weak submanifolds" and "p-submanifolds" – used in this paper. The class of p-submanifolds is the class of submanifolds with suitable tubular neighborhoods, whereas the class of weak submanifolds is the largest class of subsets that have some claim to being a "submanifold" of a certain kind. The concept of a p-submanifold is due to Melrose [56] and is central in what follows, since the class of closed p-submanifolds is the class of submanifolds with respect to which we can perform blowups. On the other hand, the concept of a weak submanifold, with which we begin, is weaker than the one considered in [56], but also arises naturally in our study. Other concepts of submanifolds are discussed in Appendix B.

2.3.1. *Submanifolds in the weakest sense: weak submanifolds.* We start the discussion with a needed notion of submanifolds, called "weak submanifolds" in order to distinguish them from other classes of submanifolds of manifolds with corners, see Appendix B. The adjective "weak" in "weak submanifolds" indicates that the class of weak submanifolds is the largest class of "submanifolds" that we consider, with the minimal requirement that a submanifold (of a manifold with corners) be also a manifold with corners in its own way for the induced structure from the ambient manifold with corners. As a first step (Definition 2.9), we introduce the class of weak submanifolds of $\mathbb{R}^n$ (or more generally of manifolds without corners or boundary). Then as a second step, see Definition 2.10, we generalize to weak submanifolds in manifolds with corners. Thus in the following definition we mainly need the case $M_1 = \mathbb{R}^n$.

Definition 2.9. Let M_1 be a *smooth* manifold (*i. e.,* without boundary or corners) of dimension n. A subset $S \subset M_1$ is a *weak submanifold* of M_1 if, and only if, any $x \in S$ is contained in the domain of a chart $\psi : U \to \Omega \subset \mathbb{R}^n$ of M_1 with $\psi(S \cap U) = (\mathbb{R}_\ell^m \times \{0\}) \cap \Omega$, for some $0 \le \ell \le m \le n$. (Hence m is the *dimension* of S at p and $n - m$ is the *codimension* of S in M_1 at p.)

A submanifold S with corners of some smooth manifold M_1 inherits from M_1 the structure of a manifold with corners. In manifolds without corners the definition of a "weak submanifolds" coincides with what often is considered as a submanifold with (boundary or) corners. The definition gets a bit more involved when we generalize to the case when M_1 is replaced with a manifold M with corners, thus obtaining a generalization of the submanifold property to the category of manifolds with corners.

Definition 2.10. A subset S of a manifold with corners M is a *weak submanifold of M* if, for every $p \in S$, there is a chart $\phi : U \to \Omega \subset \mathbb{R}_k^n$ on M, $0 \le k \le n$, such that

    (1) $p \in U$ and
    (2) $\phi(S \cap U)$ is a weak submanifold of $\mathbb{R}^n$, see Definition 2.9.

Any weak submanifold of a manifold with corners carries an induced structure of a manifold with corners, see Remark 2.12 (a) below for details. In particular, the *dimension* $\dim_p(S)$ *of S at p* is, from the definition, the dimension of $\phi(S \cap U)$ at $\phi(p)$. (Recall that the function $\dim_p(S)$, while locally constant, it is not required to be constant on a manifold with corners S.) The concept of a weak submanifold is more general than the concept of a "submanifold" considered in [56] and recalled in Definition B.1 in the Appendix. Roughly

speaking, compared to many other definitions of submanifolds in manifolds with corners in the literature, the corner structure of weak submanifolds has weaker compatibility properties with the corner structure of the ambient manifold. See Examples B.3 for examples of a weak submanifolds that are not submanifolds in the sense of Definition B.1. One can reformulate Definition 2.10 as follows.

Lemma 2.11. *The subset $S \subset M$ is a* weak *submanifold of M if, and only if, M can be extended to a smooth manifold $\widetilde{M}$ (without boundary or corners), such that S is a weak submanifold of $\widetilde{M}$. That is, for any $p \in S$ there are numbers $0 \le \ell \le m \le n = \dim M = \dim \widetilde{M}$ and a chart $\tilde{\phi} : \widetilde{U} \to \widetilde{\Omega} \subset \mathbb{R}^n$ of $\widetilde{M}$ with $p \in \widetilde{U}$ and*

$$(17) \qquad\qquad \tilde{\phi}(S \cap \widetilde{U}) \;=\; (\mathbb{R}^m_\ell \times \{0\}) \cap \widetilde{\Omega} \,.$$

In order to understand the statement of this lemma, note that ϕ in Definition 2.10 and $\tilde{\phi}$ in Lemma 2.11 have properties that are, in general, incompatible. The map ϕ turns the boundary components of M into (open subsets of) coordinate hyperfaces of $\mathbb{R}^n_k$, while the map $\tilde{\phi}$ turns the submanifold into (an open subset of) $\mathbb{R}^m_\ell \times \{0\}$. For general weak submanifolds, both properties cannot be achieved simultaneously.

Proof. On the one hand, $\left(\mathbb{R}^m_\ell \times \{0\}\right) \cap \widetilde{\Omega}$ is a weak submanifold of $\widetilde{\Omega}$. The "if" part thus follows from the fact that the property of being a weak submanifold in the sense of Definition 2.9 is invariant under restrictions to open subsets and under diffeomorphisms, which we apply to the diffeomorphism $\tilde{\phi}$. On the other hand, if we are in the situation of Definition 2.10, then $\phi(S \cap U)$ is a weak submanifold of $\mathbb{R}^n$. Let ψ be a chart for this weak submanifold in the sense of Definition 2.9. We may assume, without loss of generality, that ψ is defined on $\phi(S \cap U)$. Let $\phi_1 : U_1 \to \Omega_1$ be a smooth extension of $\phi : U \to \Omega$ to a chart of some extension $\widetilde{M}$ of M with properties as above with $U = U_1 \cap M$. Then $\tilde{\phi} := \psi \circ \phi_1$ yields a map with the properties of Equation (17) of our lemma. This yields the "only if" part and thus completes the proof. $\qquad\square$

It is clear from the construction above, that in the lemma we can choose $\widetilde{M}$ such that M is a closed subset of $\widetilde{M}$.

Remarks 2.12.

(a) Let S be a weak submanifold of M, then S is covered by charts $\tilde{\phi}\colon U \to \Omega$ of an extension $\widetilde{M}$ of M as in (17). Any $\tilde{\phi}$ yields a chart of S:

$$\psi := \tilde{\phi}|_{S \cap \widetilde{U}} : S \cap \widetilde{U} \to \left\{ x \in \mathbb{R}^{n'}_{k'} \,\middle|\, (x, 0) \in \widetilde{\Omega} \right\}.$$

The set of a such charts ψ is an atlas of S, the *induced atlas on S*. With this atlas, the set S is a manifold with corners. Thus any weak submanifold is a manifold with corners in its own.

(b) If M is manifold with corners and S a subset of the interior M_0 of M. Then S is a weak submanifold of M in the sense of Definition 2.10, if and only if it is a weak submanifold in the sense of Definition 2.9.

(c) Definition 2.9 – in which we assumed that M_1 has no corners and no boundary – is equivalent to several classical definitions in the literature. For example, a subset $S \subset M_1$ is a weak submanifold, if and only if it is a submanifold in the sense of Melrose, see Definition B.1. As discussed above and in Examples B.3, this no longer holds for submanifolds of manifolds with corners.

(d) In view of our experience with the preprint version of this article, we want to comment on a possible missunderstanding of Definition 2.10. It is a priori obvious that $\phi(S \cap U)$ is a subset of $\mathbb{R}^n_k$, so one might think, that one could or even should replace $\mathbb{R}^n$ in Definition 2.10 (2) by $\mathbb{R}^n_k$. However, this version would run into the obvious trouble that $\mathbb{R}^n_k$ is manifold with corners and weak submanifold of manifolds with corners had not yet been defined before. Thus Definition 2.10 would be circular. The possible missunderstanding is related, but more subtle. In view of item (c) one might assume that one may replace Definition 2.10 (2) by

(2′) $\phi(S \cap U)$ is a submanifold of $\mathbb{R}^n_k$ in the sense of Definition B.1,

as this property has already been defined in the literature. However, this replacement would yield a definition equivalent to Definition B.1 and thus a more restrictive notion of submanifold than ours.

(e) Melrose's more restictive definition of a submanifold given in Definition B.1 is not sufficiently general for our purposes. Indeed, the set S in Example B.3 (1) is a weak submanifold of $\mathbb{R}^2_1$, but not a submanifold in the sense of Definition B.1. Similarly, the graph blow-up is a weak submanifold of the product considered in Definition 4.17, but in general it is not a submanifold in the sense of Definition B.1, see [50] for details.

(f) Any weak submanifold S of a manifold with corners M is *locally closed*, *i.e.,* it is the intersection of a closed subset with an open subset. In order to prove this we embed M into a manifold without corners and without boundary $\widetilde{M}$ with M closed in $\widetilde{M}$, and we consider an atlas $\mathcal{A}$ of charts $\tilde{\phi} : \widetilde{U} \to \widetilde{\Omega}$ as in Lemma 2.11. Using that $(\mathbb{R}^m_\ell \times \{0\}) \cap \widetilde{\Omega}$ is closed in $\widetilde{\Omega}$, we obtain that $S \cap \widetilde{U}$ is closed in $U = \widetilde{U} \cap M$. Thus S is covered by charts of M in which S is closed, thus it is locally closed in M.

We need weak submanifolds in view of the following proposition.

Proposition 2.13. *Let N and M be manifolds with corners and let $f : N \to M$ be a smooth map that is an immersion and a homeomorphism onto its image. Then $f(N)$ is a weak submanifold of M (Definition 2.10) and $f : N \to f(N)$ is a diffeomorphism.*

Proof. By extending the charts of M, we can find an extension $\widetilde{M}$ as in Lemma 2.11, *i.e.,* M is then a submanifold with corners of codimension 0 in a manifold $\widetilde{M}$ without corners or boundary. Similar we can extend N to a manifold $\widetilde{N}$ without corners or boundary of the same dimension, and if $\widetilde{N}$ is sufficiently small, then we can find an immersion $\tilde{f} : \widetilde{N} \to \widetilde{M}$ extending f. The injectivity of f and its homeomorphism property imply (after passing to a possibly smaller $\widetilde{N} \supset N$) that the map $\tilde{f}$ is injective and satisfies the homeomorphism property.

Note that the homeomorphism property implies that we have a local statement, that is, we can restrict ourselves to small neighborhoods in N, $\widetilde{N}$, M and, respectively, $\widetilde{M}$. For $p \in N$, let $\rho : V_0 \to W_0 \subset \mathbb{R}^m_\ell$ be a chart of N containing p, $m = \dim N$, with W_0 open in $\mathbb{R}^m_\ell$, that extends to a chart $\tilde{\rho} : \widetilde{V}_0 \to \widetilde{W}_0 \subset \mathbb{R}^m$ of $\widetilde{N}$. We may extend the immersion $\tilde{f} \circ \tilde{\rho}^{-1} : \widetilde{W}_0 \to \widetilde{M}$ to a diffeomorphism $\Phi : \widetilde{W}_0 \times B_\epsilon(0, \mathbb{R}^{n-m}) \to U_0$ where U_0 is open in $\widetilde{M}$. Then $\tilde{\phi} := \Phi^{-1}$ is a chart of $\widetilde{M}$ and, if $\widetilde{V}_0$ and ϵ were chosen sufficiently small, then we may verify

$$\tilde{\phi}\big(f(N) \cap U_0\big) = (\mathbb{R}^m_\ell \times \{0\}) \cap \big(\widetilde{W}_0 \times B_\epsilon(0, \mathbb{R}^{n-m})\big).$$

The statements of the proposition then follow from Lemma 2.11. □

We obtain the following easy consequence.

Corollary 2.14. *Let N and M be manifolds with corners and $f : N \to M$ be a smooth map. If there is a smooth map $F : M \to N$ with $F \circ f = \mathrm{id}_N$, then $f(N)$ is a weak submanifold of M,* i. e., *a submanifold in the sense of Definition 2.10.*

Proof. The relation $\mathrm{id}_{T_x N} = d_{f(x)} F \circ d_x f$ implies that $d_x f : T_x N \to T_{f(x)} M$ is injective. As $F|_{f(N)}$ is continuous, f is a homeomorphism onto its image. $\square$

Proposition 2.13 can also be used to prove the following property: If P and Q are weak submanifolds of M and $P \subset Q$, then P is a weak submanifold of Q. Thus weak submanifolds have nicer categorical properties than submanifolds in the sense of Definition B.1. In the categorical language, the above property is expressed as follows: if we consider the category whose objects are manifolds with corners and whose morphisms are inclusions as a weak submanifold, then this is a *full* subcategory of the category of sets with the inclusions as morphisms. On the other hand, this property is not valid for submanifolds in the sense of Definition B.1 (see Examples B.3).

2.3.2. *Submanifolds with tubular neighborhoods: p-submanifolds.* We now recall Melrose's definition of a p-submanifold of a manifold with corners M [48, 56, 71]. In our paper, p-submanifolds are of central importance, as we blow-up manifolds with corners along closed p-submanifolds.

Recall the subsets $L_I \subset \mathbb{R}^n_k$ of Definition 2.7. After reordering the coordinates, L_I is the first factor of $\mathbb{R}^n_k \cong \mathbb{R}^d_{k-b} \times \mathbb{R}^c_b$, in the sense that L_I is mapped to $\mathbb{R}^d_{k-b} \times \{0\}$. The sets L_I are the local models for p-submanifolds [56, Definition 1.7.4]. However, in the following definition, we do not reorder the coordinates.

Definition 2.15. A subset P of a manifold with corners M is a *p-submanifold* if, for every $x \in P$, there exists a chart (U, ϕ) with $x \in U$ and $I \subset \{1, 2, \ldots, n\}$ such that

$$\phi(P \cap U) \;=\; L_I \cap \phi(U),$$

with L_I as defined in Equation (15). The number $n - |I|$ (respectively, $|I|$, respectively, $\mathrm{depth}_M(p) := |I \cap \{1, \ldots, l\}|$) will be called the *dimension* (respectively, the *codimension* of P in M at x, respectively, the *boundary depth* of P in M at x). We allow p-submanifolds Y of non-constant dimension. We define $\dim(Y)$ as the maximum of the dimensions of the connected components of Y and $\dim \emptyset = -\infty$.

Of course, this definition extends the previous definition of the boundary depth of L_I in $\mathbb{R}^n_k$ as the boundary depth of any interior point of L_I in $\mathbb{R}^n_k$. Obviously all p-submanifolds are weak submanifolds (and submanifolds in the sense of Definition B.1), and the definition of the dimension of P in x coincides with the dimension already defined above.

Remark 2.16. The codimension and the boundary depth of a p-submanifold P at $p \in P$ (introduced in Definition 2.15) are locally constant functions in p. (The dimension of P at p is also locally constant in $p \in P$, but this is true in general for weak submanifolds.) For any interior point x in P and $\epsilon > 0$ small enough, these numbers are the codimension (respectively, the the boundary depth, respectively, the dimension) of the intersection $B_\epsilon(x) \cap P$ in M. More generally: if P is a p-submanifold of M with boundary depth d on the component of $x \in P$, and if x is a (boundary) point of boundary depth e in P, then x has boundary depth $d + e$ in M. In particular, for a p-submanifold $P \subset M$, the difference of boundary depths $\mathrm{depth}_M(x) - \mathrm{depth}_P(x)$ is constant on the connected components of P.

Example 2.17. This definition of a p-submanifold comes from [56]. Note that "p" is used as an abbreviation for "product", reflecting the fact that, locally in coordinate charts, p-submanifolds are a factor of the product $\mathbb{R}_k^n \simeq \mathbb{R}_{k_1}^{n_1} \times \mathbb{R}_{k_2}^{n_2}$. More precisely, let $\ell \leq k$ and $m - \ell \leq n - k$, so that the canonical first components inclusion $j_0 : \mathbb{R}_\ell^m \to \mathbb{R}_k^n$ of Equation (13) is defined. Then $j_0(\mathbb{R}_\ell^m)$ is a p-submanifold of $\mathbb{R}_k^n$ and $j_0(\mathbb{S}_\ell^{m-1}) = \mathbb{S}^{n-1} \cap j_0(\mathbb{R}_\ell^m)$ is a p-submanifold of $\mathbb{S}_k^{n-1} = \mathbb{S}^{n-1} \cap \mathbb{R}_k^n$). In fact, it is the first factor with $n_1 = m$ and $k_1 = \ell$. A more general concept, that of an "interior binomial subvariety," was introduced and studied in [49].

Remark 2.18. Let $P \subset M$ be a p-submanifold. Then it is possible that $P \subset F$, for F a non-trivial face of M. If P is connected, then the boundary depth of P is the boundary depth of the smallest closed face F of M containing P. The earlier notion of a "submanifold of a manifolds with corners" used by some of us [2, 3, 4] is equivalent to being a p-submanifold of boundary depth 0 (*i. e.,* no connected component of P is contained in the boundary of M). They have an intrinsic definition (not using local coordinate charts).

We shall need the following lemma.

Lemma 2.19. *Let $P \subset Q \subset M$ be manifolds with corners.*
   (i) *If both P and Q are p-submanifolds of M, then P is a p-submanifold of Q.*
   (ii) *If P is a p-submanifold of Q and Q is a p-submanifold of M, then P is a p-submanifold of M.*

Proof. In order to prove (i), we consider functions $x^1, \ldots, x^\ell$ defining the p-submanifold P of codimension ℓ in M locally in a neighborhood of $x \in P$. Choose $I \subset \{1, \ldots, \ell\}$ such that $(dx^i|_p)_{i \in I}$ is a basis of $T_x^* Q$. Then in a possibly smaller neighborhood, the functions $(x^i)_{i \in I}$ define P as a p-submanifold of Q.

For (ii), we consider functions $x^1, \ldots, x^k$ locally defining P as a p-submanifold of Q. We extend these functions to locally defined functions on M. Then we choose functions $x^{k+1}, \ldots, x^l$ defining Q locally as a p-submanifold of M. Then $x^1, \ldots, x^l$ locally define P as a p-submanifold of M. $\qquad\square$

Example 2.20. The diagonal Δ_N in Example B.2 is not a p-submanifold. If N is the 2-dimensional closed disc, then with arguments analogous to Remark 4.16, the diagonal is not a p-submanifold of $N \times N$. Alternatively, one could argue using [56] (see Remark B.6).

2.3.3. *The normal bundle of p-submanifolds.* We now introduce some standard concepts that will be important in the definition of the blow-up of a manifold with corners along a p-submanifold. Recall that $T^+ M$ denotes the set of inward pointing tangent vectors to M, see Remark 2.8.

Definition 2.21. Let $P \subset M$ be a p-submanifold of the manifold with corners M.
   (i) The quotient $N^M P := TM|_P / TP$ is called the *normal bundle of P in M*.
   (ii) The image $N_+^M P$ of $T^+ M|_P$ in $N^M P$ is called the *inward pointing normal bundle of P in M*.
   (iii) The set $\mathbb{S}(N_+^M P)$ of directions in $N_+^M P$ is called the set of *inward pointing spherical normal bundle of P in M*.

Remark 2.22. As we have already noticed in Remark 2.8, $T^+ M|_P \to P$ is *not* a (locally trivial) fiber bundle over P. (Recall that $N_+^M P$ is the image of $T^+ M|_P$ in $N^M P$ via the projection $TM|_P \to N^M P := TM|_P / TP$.) Then the projection map $N_+^M P \to P$ does define a (locally trivial) fiber bundle structure over P on $N_+^M P$, precisely because P is a

p-submanifold. The fibers of this bundle are modeled on $\mathbb{R}_r^\ell$, where ℓ is the codimension of P and r is its boundary depth. Thus the name of "inward pointing normal bundle of P in M" for $N_+^M P$ is justified. Consequently, $\mathbb{S}(N_+^M P)$, the inward pointing spherical normal bundle of P in M, is also a (locally trivial) fiber bundle over P with projection $\mathbb{S}(N_+^M P) \to P$. If M is endowed with a Riemannian metric, then, we can endow $N_+^M P \to P$ with the quotient metric, and hence we can of course identify $\mathbb{S}(N_+^M P)$ with the set of unit vectors in $N_+^M P$. In particular, if, at $p \in P$, we have $\dim_p(P) = \dim_p(M)$, then the total space of this fiber bundle is $\mathbb{S}_p(N_+^M P) = \emptyset$.

We complete this section with a related remark. We notice, in particular, the existence of suitable "tubular neighborhoods" for p-submanifolds.

Remark 2.23. Let $P \subset M$ be a p-submanifold in the manifold with corners M. If M is compact, then P has a neighborhood $V_P \subset M$ such that V_P is diffeomorphic to the closed cone $N_+^M P$ via a diffeomorphism that sends P to the zero section of $N_+^M P \to M$ and induces the identity at the level of normal bundles. This was proved in [56, Proposition 2.10.1], under the additional assumption that P be closed. Moreover, the condition that M be compact or that P be closed is not necessary, provided the right definition of tubular neighborhood is chosen (as neighborhood that is a disc bundle over P, which, in general, will not be the set of points of distance $< \epsilon$ for some Riemannian metric and some small ϵ). Then $N_+^M P$ is a cone with corners in $N^M P$. Generalizing Example 2.6, we obtain that all of the subsets $N^M P$, $N_+^M P$, and $S^+(N^M P)$ of TM introduced in the last definition are manifolds with corners. This is because the property of being a manifold with corners is a local property and the product of manifolds with corners is again a manifold with corners.

We finish the discussion of p-submanifolds by stressing that we allow the different connected components of a p-submanifold to have *different* dimensions. This is convenient when considering intersections and when defining blow-ups.

3. THE BLOW-UP FOR MANIFOLDS WITH CORNERS

We now recall and study the blow-up of a manifold M with corners by a *closed* p-submanifold. Our definition is the same as the one in [1, 48, 56].

3.1. **Definition of the blow-up and its smooth structure.** To define the blow-up, we first define the underlying set and then we will define its smooth structure using as a model the case of euclidean spaces. The disjoint union of two subsets A and B will be typically denoted $A \sqcup B$, as usual.

3.1.1. *Definition of the blow-up as a set.* We now define the underlying set of the blow-up $[M : P]$ of a manifold with corners M with respect to a closed p-submanifold P by replacing P with the inward spherical normal bundle $\mathbb{S}(N_+^M P)$ of P in M using the disjoint union $\sqcup$.

Definition 3.1. Let M be a manifold with corners and P be a *closed* p-submanifold of M. Let $\mathbb{S}(N_+^M P)$ be the inward pointing spherical normal bundle of P in M (Definition 2.21). As a set, we define the *blow-up $[M : P]$ of M along P* (or *with respect to P*) as the disjoint union

$$[M : P] := (M \setminus P) \sqcup \mathbb{S}(N_+^M P).$$

The blow-down map $\beta = \beta_{M,P} : [M : P] \to M$ is defined as the identity map on $M \setminus P$ and as the fiber bundle projection $\mathbb{S}(N_+^M P) \to P$ on the complement.

Let us comment now on the definition of the blow-up.

Remark 3.2. The blow-up $[M : P]$ is therefore not defined if P is not closed, but we allow P to consist of the disjoint union of several closed, connected p-submanifolds of M of possibly different dimensions. Also, we are allowing the submanifold P to be empty or to have components of the same dimension as M, which will be convenient when considering iterated blow-ups, since it is not always possible to arrange a family of submanifolds in a strictly increasing order of dimensions. The components of P of the same dimension with M will, of course, be connected components of M (since we have assumed that P is a closed p-submanifold of M). If $P \neq \emptyset$ is a union of connected components of M, then $[M : P] = M \smallsetminus P$ and the blow-down map $\beta_{M,P} : [M : P] \to M$ is not surjective in this case (see Remark 2.22). In particular $[M : M] = \emptyset$, and hence the case $P = \emptyset$ is also needed when considering iterated blow-ups. In that case, $[M : \emptyset] = M$.

A general approach to smooth structures on the blow-up is contained in [49]. Here we recall an approach that suffices for our needs. We begin with the case of open subsets of a model space $\mathbb{R}^n_k$.

3.1.2. *The blow-up of the local models.* We now discuss various issues concerning the blow-up in local coordinates. In the following, let I_j, $j = 1, 2, \ldots n$, denote either $\mathbb{R}$ or $[0, \infty)$. We will write $N_1 \cong N_2$ if N_1 is a p-submanifold of $I_1 \times I_2 \times \cdots \times I_n \subset \mathbb{R}^n$ and if there is a permutation σ of the components of $\mathbb{R}^n$ that induces a diffeomorphism from N_1 to the p-submanifold N_2 of $I_{\sigma(1)} \times I_{\sigma(2)} \times \cdots \times I_{\sigma(n)} \subset \mathbb{R}^n$. By contrast, when we write $N_1 \simeq N_2$, we will merely state that the indicated manifolds are diffeomorphic, without including further information on the diffeomorphism. In particular, $N_1 \cong N_2$ implies $N_1 \simeq N_2$. To start with, the blow-up $[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} : \mathbb{R}^n_k \times \{0\}]$ of $\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \cong \mathbb{R}^{n+n'}_{k+k'}$ along its p-submanifold $\mathbb{R}^n_k \times \{0\} = \mathbb{R}^n_k \times \{0_{\mathbb{R}^{n'}}\}$ is, by Definition 3.1, the set

$$
\begin{aligned}
(18) \quad [\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} : \mathbb{R}^n_k \times \{0\}] \ &:= \ \left(\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \smallsetminus \mathbb{R}^n_k \times \{0\} \right) \sqcup \mathbb{R}^n_k \times \mathbb{S}^{n'-1}_{k'} \\
&= \ \mathbb{R}^n_k \times \left(\mathbb{S}^{n'-1}_{k'} \sqcup \left(\mathbb{R}^{n'}_{k'} \smallsetminus \{0\} \right) \right).
\end{aligned}
$$

Let us consider the map

$$
\begin{aligned}
(19) \quad \kappa : \mathbb{R}^n_k \times \mathbb{S}^{n'-1}_{k'} \times [0, \infty) \ &\to \ \mathbb{R}^n_k \times \left(\mathbb{S}^{n'-1}_{k'} \sqcup \left(\mathbb{R}^{n'}_{k'} \smallsetminus \{0\} \right) \right), \\
\kappa(x, \xi, r) \ &:= \ \begin{cases} (x, \xi) \in \mathbb{R}^n_k \times \mathbb{S}^{n'-1}_{k'} & \text{if } r = 0 \\ (x, r\xi) \in \mathbb{R}^n_k \times (\mathbb{R}^{n'}_{k'} \smallsetminus \{0\}) & \text{if } r > 0. \end{cases}
\end{aligned}
$$

The map κ is immediately seen to be a bijection and we will use it to endow $[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} : \mathbb{R}^n_k \times \{0\}]$ with the structure of a manifold with corners induced from $\mathbb{R}^n_k \times \mathbb{S}^{n'-1}_{k'} \times [0, \infty)$. Under this diffeomorphism, the blow-down map becomes

$$
(20) \qquad \beta : \mathbb{R}^n_k \times \mathbb{S}^{n'-1}_{k'} \times [0, \infty) \to \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'}, \quad \beta(x, \xi, r) := (x, r\xi).
$$

The blown-up space $[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} : \mathbb{R}^n_k \times \{0\}]$ is thus a space of "generalized spherical coordinates."

If $U \subset \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'}$ is an open subset, we endow

$$
(21) \qquad [U : U \cap (\mathbb{R}^n_k \times \{0\})] = \beta^{-1}(U) \subset [\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} : \mathbb{R}^n_k \times \{0\}]
$$

with the induced structure of a manifold with corners.

3.1.3. *The topology and smooth structure of the blow-up.* The following lemmas will allow us to define a manifolds with corners structure on blow-ups.

Lemma 3.3. *Let $P_i \subset M_i$, $i = 1, 2$, be closed p-submanifolds and let $\phi : M_1 \to M_2$ be a diffeomorphism such that $\phi(P_1) = P_2$. Then there exists a unique map $\phi^\beta : [M_1 : P_1] \to [M_2 : P_2]$ that is bijective and makes the following diagram commute*

$$
\begin{array}{ccc}
[M_1 : P_1] & \xrightarrow{\phi^\beta} & [M_2 : P_2] \\
\beta_{M_1,P_1} \downarrow & & \downarrow \beta_{M_2,P_2} \\
M_1 & \xrightarrow{\phi} & M_2.
\end{array}
$$

This construction is functorial, in the sense that $(\phi \circ \psi)^\beta = \phi^\beta \circ \psi^\beta$. If M_i are open subsets of $\mathbb{R}^n_k$, then ϕ^β is a diffeomorphism.

Proof. The existence, uniqueness, and the functorial character of ϕ^β follows from the definition of the blow-up. The fact that ϕ^β is smooth if M_i are open subsets of the model space $\mathbb{R}^n_k$ is the content of Lemma 2.2 of [1]. □

Recall the maps ϕ^β of Lemma 3.3.

Lemma 3.4. *Let $\mathcal{A} = \{(U_a, \phi_a) \,|\, a \in A\}$ be an* atlas *on a manifold with corners M (Definition 2.4). Let $P \subset M$ be a closed p-submanifold and $\beta = \beta_{M,P} : [M : P] \to M$ be the blow-down map. We endow $[M : P]$ with the smallest topology that makes all the maps ϕ_a^β, $a \in A$, continuous (ϕ_a^β is defined on $\beta^{-1}(U_a)$, see Lemma 3.3). Then*

$$
\beta^*(\mathcal{A}) := \{(\beta^{-1}(U_a), \phi_a^\beta) \,|\, a \in A\}
$$

is an atlas on $[M : P]$. If we take another atlas $\mathcal{A}'$ of M that is compatible with $\mathcal{A}$, then $\beta^*(\mathcal{A})$ and $\beta^*(\mathcal{A}')$ will be compatible atlases on $[M : P]$.

Proof. This follows from Equation (21) and Lemma 3.3. □

Lemma 3.4 thus yields the desired smooth structure on $[M : P]$ that is moreover canonical (independent of any choices).

Definition 3.5. Let M be a manifold with corners and $P \subset M$ be a closed p-submanifold. We endow $[M : P]$ with the smooth structure defined by the atlas $\beta^*(\mathcal{A})$ obtained from Lemma 3.4, for any atlas $\mathcal{A}$ on M.

The following is a consequence of the definition.

Corollary 3.6. *With the notations of Definition 3.5, $M \smallsetminus P$ is dense in $[M : P]$.*

This result remains true even if P is a union of connected components of M, in which case $[M : P] = M \smallsetminus P$. The smooth structure on $[M : P]$ is natural in the following strong sense.

Proposition 3.7. *With the notation of Lemma 3.3, we have that the map ϕ^β is a diffeomorphism (in general, not just in the case of open subsets of Euclidean spaces).*

Proof. If $\mathcal{A}$ is an atlas on M_2, then the pull-back of $\beta^*(\mathcal{A})$ to $[M_1 : P_1]$ is an atlas. □

The blow-up $[M : P]$ of a manifold with corners is thus again a manifold with corners.

3.2. **Exploiting the local structure of the blow-up.** The local character of the definition of the smooth structure of the blow-up $[M : P]$ of the manifold with corners M along a closed p-submanifold P means that most of the proofs involving blow-ups can be conveniently treated by first treating the model case $P := \mathbb{R}_k^n = \mathbb{R}_k^n \times \{0\} \subset \mathbb{R}_k^n \times \mathbb{R}_{k'}^{n'} = M$. To simplify notation, we shall often omit factors of the form $\{0\}$ when there is no danger of confusion. This is the case with the following results.

3.2.1. *The blow-down map is proper.* We shall need to prove that certain maps are closed. This will be conveniently done by proving that they are proper, since a proper map between manifolds with corners is closed. In particular, we will show that the blow-down map is proper.

Let $f : X \to Y$ be a continuous map between two Hausdorff spaces. Recall that f is called *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset Y$. For instance, the map β of Equation (20) is immediately seen to be proper.

Corollary 3.8. *Let P be a closed p-submanifold of a manifold with corners M. The blow-down map $\beta_{M,P} : [M : P] \to M$ is proper.*

Proof. Using Lemma A.3 from the Appendix, we see that we can treat the problem in local coordinates. Then, in local coordinates, the blow-down map is given by Equation (20), which is a proper map, as we have already pointed out. $\square$

3.2.2. *Blow-ups and products.* We have a simple, convenient behavior of the blow-up with respect to products.

Lemma 3.9. *Let M and M_1 be two manifolds with corners and P be a closed p-submanifold of M. Then $P \times M_1$ is a closed p-submanifold of $M \times M_1$ and there exists a canonical diffeomorphism $[M \times M_1 : P \times M_1] \simeq [M : P] \times M_1$ such that the following diagram commutes:*

(22)
$$
\begin{array}{ccc}
[M \times M_1 : P \times M_1] & \xrightarrow{\ \simeq\ } & [M : P] \times M_1 \\
{\scriptstyle \beta_{M \times M_1, P \times M_1}} \big\downarrow & & \big\downarrow {\scriptstyle \beta_{M,P} \times \mathrm{id}} \\
M \times M_1 & \xrightarrow{\ \mathrm{id}\ } & M \times M_1 \, .
\end{array}
$$

Proof. Since the result is a local one and P is a closed p-submanifold of M, it is enough to treat the case

$$
\begin{aligned}
M & := \mathbb{R}_{k_m}^m \times \mathbb{R}_{k_p}^p \\
P & := \{0_{\mathbb{R}^m}\} \times \mathbb{R}_{k_p}^p \subset M \\
M_1 & := \mathbb{R}_{k_l}^l \, .
\end{aligned}
$$

In this local treatment, we will again write $\cong$ to stress that a given diffeomorphism is given by a permutation of coordinates, as in Equation (14).

With this choice, we see that $P \times M_1$ is a closed p-submanifold of $M \times M_1$. We have natural diffeomorphisms with the first one being obtained from the definition of the blow-up, Definition 3.1, and the last being induced by suitable permutations of coordinates

$$
\begin{aligned}
[M \times M_1 : P \times M_1] &= [\mathbb{R}_{k_m}^m \times \mathbb{R}_{k_p}^p \times \mathbb{R}_{k_l}^l : \{0_{\mathbb{R}^m}\} \times \mathbb{R}_{k_p}^p \times \mathbb{R}_{k_l}^l] \\
&= \mathbb{S}_{k_m}^{m-1} \times \mathbb{R}_{k_p}^p \times \mathbb{R}_{k_l}^l \sqcup \left((\mathbb{R}_{k_m}^m \times \mathbb{R}_{k_p}^p \times \mathbb{R}_{k_l}^l) \smallsetminus (\{0_{\mathbb{R}^m}\} \times \mathbb{R}_{k_p}^p \times \mathbb{R}_{k_l}^l) \right) \\
&\simeq \mathbb{S}_{k_m}^{m-1} \times [0, \infty) \times \mathbb{R}_{k_p}^p \times \mathbb{R}_{k_l}^l \cong \mathbb{S}_{k_m}^{m-1} \times \mathbb{R}_{k_p+k_l+1}^{p+l+1}
\end{aligned}
$$

and

$$\begin{aligned}
[M : P] \; &= \; [\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} : \{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p}] \\
&= \; \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^p_{k_p} \sqcup \left((\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p}) \smallsetminus (\{0_{\mathbb{R}^m}\} \times \mathbb{R}^p_{k_p}) \right) \\
&\simeq \; \mathbb{S}^{m-1}_{k_m} \times [0,\infty) \times \mathbb{R}^p_{k_p} \; \cong \; \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1} \,.
\end{aligned}$$

The desired diffeomorphism $[M \times M_1 : P \times M_1] \xrightarrow{\simeq} [M : P] \times M_1$ is then induced by the above diffeomorphisms and the canonical permutation of coordinates diffeomorphism $\mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1} \times \mathbb{R}^l_{k_l} \xrightarrow{\cong} \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+l+1}_{k_p+k_l+1}$ of Equation (14). $\qquad\square$

The functoriality property of Lemma 3.3 and Lemma 3.9 then gives the following result.

Proposition 3.10. *Let G be a Lie group acting smoothly on M (that is, such that the action map $G \times M \to M$ is smooth). Let $P \subset M$ be a closed p-submanifold such that $g(P) = P$ for all $g \in G$. Then the action of G lifts to a smooth action of G on $[M : P]$.*

Proof. We extend the action map to a diffeomorphism

$$a : G \times M \to G \times M \,, \quad (g, x) \mapsto (g, gx) \,.$$

Note that $G \times P$ is a closed p-submanifold of $G \times M$. Thus our functorial property implies that this map lifts to a map

$$[G \times M : G \times P] \to [G \times M : G \times P] \,.$$

Using the natural diffeomorphism $[G \times M : G \times P] \simeq G \times [M : P]$ of Lemma 3.9, we obtain a smooth map

$$\hat{a} : G \times [M : P] \to G \times [M : P].$$

Thus the second component $\hat{a}_2$ defines a smooth map $G \times [M : P] \to [M : P]$. It satisfies all axioms of an action since it is an extension of the action map for the action of G on $M \smallsetminus P$ and since $M \smallsetminus P$ is dense in $[M : P]$. $\qquad\square$

3.3. **Cleanly intersecting families and liftings.** In order to avoid complications, we shall consider the blow-up with respect to "cleanly intersecting families" of closed p-submanifolds and "clean semilattices," some concepts that we recall below.

3.3.1. *Clean intersections.* We continue to exploit the local structure of the blow-up. Recall the following standard definition.

Definition 3.11. Let M be a manifold with corners and $X_1, X_2, \ldots, X_k \subset M$ be p-submanifolds. We shall say that $X_1, X_2, \ldots, X_k$ *have a clean intersection* or that they *intersect cleanly* if

    (i) $Y := X_1 \cap X_2 \cap \ldots \cap X_k$ is a p-submanifold of M (possibly empty),
    (ii) for all $x \in Y$, $T_x Y = T_x X_1 \cap T_x X_2 \cap \ldots \cap T_x X_k$.

We consider conditions (i) and (ii) of Definition 3.11 to be automatically satisfied if $Y := X_1 \cap X_2 \cap \ldots \cap X_k = \emptyset$. Similar conditions appear in [1], Definition 2.7, where they were used to define a *weakly transversal family* of connected submanifolds with corners. We shall need also the notion of a "cleanly intersecting family" (Definition 4.10), which roughly states that every subfamily intersects cleanly.

Lemma 3.12. *Let P and Q be p-submanifolds of M intersecting cleanly. Then $P \cap Q$ is a p-submanifold of Q (and also for P).*

Proof. According Definition 3.11 (i) $P \cap Q$ is a p-submanifold of M. Then Lemma 2.19 (i) states that $P \cap Q$ is also a p-submanifold of Q. $\qquad\square$

3.3.2. *Liftings of p-submanifolds to blow-ups.* We now consider the lifting of suitable p-submanifolds in M to $[M : P]$ as in [49, 56]. The local model for such lifts is given by the following lemma.

Lemma 3.13. *Let $k' \leq k''$ and $n' - k' \leq n'' - k''$, so that the canonical first components inclusion $j_0 : \mathbb{R}^{n'}_{k'} \to \mathbb{R}^{n''}_{k''}$ of Equation* (13) *is defined, and let $j := (\mathrm{id} \times j_0, 0) : \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \to \mathbb{R}^n_k \times \mathbb{R}^{n''}_{k''} \times \mathbb{R}^p_\ell$. Then there is a unique smooth map j^β such that the diagram*

$$
\begin{array}{ccc}
[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} : \mathbb{R}^n_k \times \{0\}] & \xrightarrow{\ j^\beta\ } & [\mathbb{R}^n_k \times \mathbb{R}^{n''}_{k''} \times \mathbb{R}^p_\ell : \mathbb{R}^n_k \times \{0\} \times \mathbb{R}^p_\ell] \\
{\scriptstyle \beta_{\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'}, \mathbb{R}^n_k \times \{0\}}} \Big\downarrow & & \Big\downarrow {\scriptstyle \beta_{\mathbb{R}^n_k \times \mathbb{R}^{n''}_{k''} \times \mathbb{R}^p_\ell, \mathbb{R}^n_k \times \{0\} \times \mathbb{R}^p_\ell}} \\
\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} & \xrightarrow{\quad j \quad} & \mathbb{R}^n_k \times \mathbb{R}^{n''}_{k''} \times \mathbb{R}^p_\ell\,.
\end{array}
$$

commutes ($0 \leq \ell \leq p$). Moreover, j^β is a diffeomorphism onto its image, which is a closed p-submanifold of $[\mathbb{R}^n_k \times \mathbb{R}^{n''}_{k''} : \mathbb{R}^n_k \times \{0\}]$.

The definition of the map j^β extends the definition in Lemma 3.3.

Proof. If we can prove the result for $p = \ell = 0$, then Lemma 3.9 will give the result in general. Clearly, j_0 is a particular case of the maps j (more precisely the case $n = k = 0$), but we will treat it first. Let us first notice that in the statement, we have not specified to which sets the various neutral elements belong. For instance, the desired map j_0^β acts as $j_0^\beta : [\mathbb{R}^{n'}_{k'} : \{0_{\mathbb{R}^{n'}}\}] \to [\mathbb{R}^{n''}_{k''} : \{0_{\mathbb{R}^{n''}}\}]$. However, in what follows, we will drop the indices of the neutral elements, as there is no danger of confusion. To define j_0^β and to obtain its properties, we shall use the definition of the blow-up in local coordinates and the setting of Example 2.17 to define

$$
j_0^\beta : [\mathbb{R}^{n'}_{k'} : \{0\}] \simeq \mathbb{S}^{n'-1}_{k'} \times [0, \infty) \xrightarrow{\ i \times \mathrm{id}\ } \mathbb{S}^{n''-1}_{k''} \times [0, \infty) \simeq [\mathbb{R}^{n''}_{k''} : \{0\}]\,,
$$

where $i : \mathbb{S}^{n'-1}_{k'} \to \mathbb{S}^{n''-1}_{k''}$ is the restriction of j_0. With this definition, we obtain that j_0^β fits into the commutative diagram

$$
\begin{array}{ccc}
[\mathbb{R}^{n'}_{k'} : \{0\}] & \xrightarrow{\ j_0^\beta\ } & [\mathbb{R}^{n''}_{k''} : \{0\}] \\
{\scriptstyle \beta_{\mathbb{R}^{n'}_{k'}, \{0\}}} \Big\downarrow & & \Big\downarrow {\scriptstyle \beta_{\mathbb{R}^{n''}_{k''}, \{0\}}} \\
\mathbb{R}^{n'}_{k'} & \xrightarrow{\ j_0\ } & \mathbb{R}^{n''}_{k''}\,,
\end{array}
$$

As in Example 2.17, j_0^β is seen to be a diffeomorphism onto its image, which is a closed p-submanifold.

Coming back to the general case, let $\mathrm{id} : \mathbb{R}^n_k \to \mathbb{R}^n_k$ be the identity map. We then let j^β be the composition of $\mathrm{id} \times j_0^\beta : \mathbb{R}^n_k \times [\mathbb{R}^{n'}_{k'} : \{0\}] \to \mathbb{R}^n_k \times [\mathbb{R}^{n''}_{k''} : \{0\}]$ with the canonical diffeomorphisms $\mathbb{R}^n_k \times [\mathbb{R}^{n'}_{k'} : \{0\}] \simeq [\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} : \mathbb{R}^n_k \times \{0\}]$ and $\mathbb{R}^n_k \times [\mathbb{R}^{n''}_{k''} : \{0\}] \simeq [\mathbb{R}^n_k \times \mathbb{R}^{n''}_{k''} : \mathbb{R}^n_k \times \{0\}]$ of Lemma 3.9. Then j^β is smooth, by its definition. Lemma 3.9 also gives the commutativity of the diagram and proves that the image of j^β is a p-submanifold, and hence that j^β is a diffeomorphism onto its image. $\qquad\square$

We have the following result on the blow-up of p-submanifolds, due, in part, to Melrose [56, Chapter 5, Section 7]. A proof in a slightly less general setting can be found also in

Proposition 2.4 of [1]. For a p-submanifold $P \subset M$, recall the definition of $\mathbb{S}(N_+^M P)$, the *inward pointing normal bundle of P in M* from Definition 2.21.

Proposition 3.14. *Let P and Q be closed p-submanifolds of M intersecting cleanly. Let $j : Q \to M$ be the inclusion. There exists a unique continuous map*

$$j^\beta : [Q : P \cap Q] \to [M : P]$$

such that $\beta_{M,P} \circ j^\beta = j \circ \beta_{Q,P \cap Q}$. The map j^β is an injective immersion and its image is a closed p-submanifold. Moreover

$$\overline{\beta_{M,P}^{-1}(Q \smallsetminus P)} \;=\; j^\beta([Q : P \cap Q]).$$

In the following we will identify $[Q : P \cap Q]$ with $j^\beta([Q : P \cap Q])$.

Proof. Let $P_0 := P \cap Q$, to simplify the notation. To define the map

$$j^\beta : [Q : P_0] \;:=\; (Q \smallsetminus P_0) \sqcup \mathbb{S}(N_+^Q P_0) \to (M \smallsetminus P) \sqcup \mathbb{S}(N_+^M P) \;=:\; [M : P],$$

it is enough to define it on $Q \smallsetminus P_0$ and on $\mathbb{S}(N_+^Q P_0)$. First, on $Q \smallsetminus P_0$, we let j^β to be the inclusion $Q \smallsetminus P_0 := Q \smallsetminus (P \cap Q) \to M \smallsetminus P$. The inclusion of Q into M also induces an inclusion $TQ \to TM$, extending the inclusion $TP_0 \to TP$. Since $TP_0 = TP \cap TQ$, by the assumption that P and Q intersect cleanly, we can pass to quotients to obtain an vector bundle map

$$\psi : N^Q P_0 \;:=\; TQ|_{P_0}/TP_0 \;=\; TQ|_{P_0}/(TP \cap TQ) \;\to\; TM|_P/TP \;=:\; N^M P,$$

which is injective, immersive, and a homeomorphism onto its image, which is, moreover, a closed subset of $N^M P$. Next, the map ψ restricts to an embedding $\mathbb{S}(N_+^Q P) \to \mathbb{S}(N_+^M P)$, which defines j^β on $\mathbb{S}(N_+^Q P)$. This completes the definition of j^β. This definition shows right away that j^β is injective and that it satisfies the relation $\beta_{M,P} \circ j^\beta = j \circ \beta_{Q,P_0}$.

We next want to show that j^β is smooth and that $j^\beta([Q : P \cap Q])$ is a p-submanifold of $[M : P]$. The locality of the blow-up and from Lemma 3.9 it follows that it is sufficient to prove this in the case $Q := \mathbb{R}_k^n \times \mathbb{R}_{k'}^{n'} \times \{0\} \times \{0\} \subset M := \mathbb{R}_k^n \times \mathbb{R}_{k'}^{n'} \times \mathbb{R}_{k''}^{n''} \times \mathbb{R}_\ell^p$ and $P := \mathbb{R}_k^n \times \{0\} \times \{0\} \times \mathbb{R}_\ell^p$. Lemma 3.13 (a standard calculation in local coordinates, but note that n'' in that Lemma is $n' + n''$ in our definition) then gives the desired statement (that j^β is smooth and that $j^\beta([Q : P_0])$ is a p-submanifold, where, we recall, $P_0 = P \cap Q$). To prove that $j^\beta([Q : P_0])$ is closed in $[M : P]$, we can use again Lemma 3.9. Indeed, let $x_n \in j^\beta([Q : P_0])$ be a sequence convergent to some $y \in [M : P]$. We want to show that $y \in j^\beta([Q : P_0])$. Let $z := \beta_{M,P}(y)$, which is the limit of $\beta_{M,P}(x_n)$. Then the result follows using local coordinates around z compatible with Lemma 3.13.

We now come back to the general case. Since j^β is smooth, it is also an immersion, by its definition, since we have already noticed that it is an immersion on the two sets defining $[Q : P \cap Q]$. (An alternative argument would be to use the local description of the blow-up in terms of half-spaces as in [1].) It follows that j^β is a diffeomorphism onto its image. We also obtain that j^β is unique, since it is continuous (even differentiable!) and since $Q \smallsetminus (P \cap Q)$ is dense in $[Q : P \cap Q]$, by Corollary 3.6. The equality $\overline{\beta_{M,P}^{-1}(Q \smallsetminus P)} = j^\beta([Q : P \cap Q])$ follows since the right hand side is a closed set containing $Q \smallsetminus (P \cap Q)$. $\quad\square$

Definition 3.15. Let P be a closed p-submanifold of M and Q be a closed subset of M. The *lifting* $\beta_{M,P}^*(Q)$ of Q in $[M : P]$ is defined by

$$\beta_{M,P}^*(Q) \;:=\; \overline{\beta_{M,P}^{-1}(Q \smallsetminus P)}.$$

(The closure is in $[M : P]$.)

Remark 3.16. Of course, in general, $\beta^*_{M,P}(Q)$ will not be a submanifold of some sort (of $[M : P]$), even if Q is one. However, if Q is a closed p-submanifold of M and P and Q intersect cleanly, then $\beta^*_{M,P}(Q) \simeq j^\beta([Q : P \cap Q])$ and will hence be a p-submanifold of M, by Proposition 3.14. Also, it should be pointed out that, if $Q \subset P$, the definition above of $\beta^*_{M,P}(Q)$ differs from Melrose's one in [56, Chapter 5, Section 7]. More precisely, if $Q \subset P$, our definition is such that $\beta^*_{M,P}(Q) = \emptyset$, whereas it is $\beta^{-1}_{M,P}(Q)$ in Melrose's unpublished book. In part for this reason, we will avoid considering the case $Q \subset P$ when dealing with $\beta^*_{M,P}(Q)$. A first advantage of our definition is that it preserves the inclusions. Another advantage of our definition is that it is local in the sense that, for any open subset $U \subset M$, we have

$$\beta^*_{U,P\cap U}(Q \cap U) \;=\; \beta^*_{M,P}(Q) \cap \beta^{-1}_{M,P}(U)\,.$$

Moreover, in Melrose's definition, locality may fail if Q is not connected. However, with our definition, in general $\big[[M : Q] : P\big] \neq \big[[M : P] : Q\big]$ if $Q \subsetneq P$.

4. The iterated and the graph blow-ups

We introduce, study, and compare in this section two types of blow-ups with respect to more than one submanifold: the *iterated blow-up* and the *graph blow-up*. The graph blow-up $\{M : \mathcal{P}\}$ has the advantage that it is defined in great generality and is obviously independent of the order on the family of closed p-submanifolds $\mathcal{P}$, up to an isomorphism. However, it is not clear whether it has a smooth structure. The iterated blow-up, on the other hand, comes with a smooth structure, but may not be always defined and, at least in our approach, depends on the order on the elements of the family $\mathcal{P}$.

4.1. **Definition of the iterated blow-up.** Recall the definition of the lifting $\beta^*(Q) = \beta^*_{M,P}(Q) := \overline{Q \smallsetminus P} \subset [M : P]$ (closure in $[M : P]$), Definition 3.15. We fix a manifold with corners M. We now introduce the *iterated version of the blow-up*; it is different from the one in [48, 56] since we are using a different type of pull-back β^*.

Definition 4.1. Let $\mathcal{P} := (P_i)_{i=1}^k$, $P_i \subset M$, be a k-tuple of subsets of M with P_1 a closed p-submanifold of M, let $\beta_1 := \beta_{M,P_1} : [M : P_1] \to M$ be the associated blow-down map, and let $\mathcal{P}' := \big(\beta^*_1(P_2), \beta^*_1(P_2), \ldots, \beta^*_1(P_k)\big)$, which is a $(k-1)$-tuple. If $k = 1$ or if $\big[[M : P_1] : \mathcal{P}'\big]$ is defined, then we define by induction on k the *iterated blow-up* $[M : \mathcal{P}]$ *of M with respect to* or *along $\mathcal{P}$* by

$$[M : \mathcal{P}] \;=\; [M : (P_i)_{i=1}^k] \;:=\; \begin{cases} [M : P_1] & \text{if } k = 1\,, \\ \big[[M : P_1] : \mathcal{P}'\big] & \text{if } k > 1\,. \end{cases}$$

Let us now discuss this definition.

Remark 4.2. The following comments should be compared with the analogous ones for the graph-blow-up (Remark 4.18).

(1) We can remove from the k-tuple $\mathcal{P}$ any element P_j for which there exists a *larger* element P_i preceding it (i.e., there exists $i < j$ with $P_i \supset P_j$) without affecting the resulting blown-up space. To see that, we notice that $\beta_{M,P}(Q) = \emptyset$ if $Q \subset P$ and $[Q : \emptyset] = Q$.

(2) Similarly, we can also remove any occurence of the empty set from the k-tuple without affecting the resulting blow-up. In particular, removing or adding repetitions to a k-tuple $\mathcal{P}$ or removing or adding empty sets to the k-tuple will not affect

whether $[M : \mathcal{P}]$ is defined or not. (When removing repetitions, we always keep the first element in the repeated sequence and remove only the ones following it.)

(3) Let us say that the k-tuple $\mathcal{P}$ is *degenerate* if it contains repetitions. Otherwise, we shall say that it is *non-degenerate*. Thus, we can always replace our k-tuple $\mathcal{P}$ with non-degenerate one without changing the result of the iterated blow-up (in particular, not changing the whether it is defined or not). We then notice that $(k-1)$-tuple $\mathcal{P}'$ may be degenerate even if $\mathcal{P}$ is non-degenerate. An example is provided by the triple $\mathcal{P} := (P, Q, P \cup Q)$, where P and Q are *disjoint* p-sub-manifolds of M.

(4) We also notice that giving a k-tuple without repetitions of p-submanifolds of M is equivalent to giving a totally ordered finite set (or family) of p-submanifolds of M and that, when considering blow-ups, it is enough to consider only such totally ordered sets.

(5) Finally, in [48], Kottke has introduced a *size order* as one that satisfies

$$P_i \subsetneq P_j \ \Rightarrow i < j$$

and used it to define his version of iterated blow-up. (His version of iterated blow-up, however, is different from ours since he considered a different pull-back map β^*.) The above discussion shows that, given a k-tuple $\mathcal{P}$, there always exists another j-tuple $\mathcal{P}_0$, $j \leq k$, that is size-ordered and such that $[M : \mathcal{P}] = [M : \mathcal{P}_0]$ (this statement subsumes the fact that if one of these blow-ups is defined, then so is the other).

We shall also write

$$[M : \mathcal{P}] \ = \ [M : (P_i)_{i=1}^k] \ =: \ [M : P_1, P_2, \ldots, P_k] \,,$$

and hence, using the pull-back by the map β_1, we have

$$[M : P_1, P_2, \ldots, P_k] \ := \ \big[[M : P_1] : \beta_1^*(P_2), \ldots, \beta_1^*(P_k)\big] \,.$$

We generalize this relation in the following remark.

Remark 4.3. Let us define by induction first $\beta_1 := \beta_{M,P_1} : [M : P_1] \to M$, and $\gamma_1 := \beta_1^*$ and then, for $k \geq 2$,

$$\beta_k \ := \ \beta_{[M,P_1,P_2,\ldots,P_{k-1}],\gamma_{k-1}^*(P_k)} : [M : P_1, P_2, \ldots, P_k] \to [M : P_1, P_2, \ldots, P_{k-1}]$$

and $\gamma_k := \beta_k^* \circ \gamma_{k-1} = \beta_k^* \circ \ldots \circ \beta_1^*$. (In particular, $\beta_2 := \beta_{[M:P_1],\beta_1^*(P_2)} : [[M : P_1] : \beta_1^*(P_2)] \to [M : P_1]$.) We then have

$$
\begin{aligned}
[M : P_1, P_2, \ldots, P_j] \ &= \ \big[[M : P_1] : \gamma_1(P_2), \ldots, \gamma_1(P_j)\big] \\
&= \ \Big[[[M : P_1] : \gamma_1(P_2)] : \gamma_2(P_3), \ldots, \gamma_2(P_j)\Big] \\
&= \ \ldots \\
&= \ [[\ldots [[[M : P_1] : \gamma_1(P_2)] : \gamma_2(P_3)] \ldots] : \gamma_{j-1}(P_j)] \,.
\end{aligned}
$$

Note that $[M : P_1, P_2, \ldots, P_j]$ is always defined if $j = 1$. Then the condition that the iterated blow-up $[M : P_1, P_2, \ldots, P_j]$ be defined can then be formulated by induction as follows:

 (i) the iterated blow-up $[M : P_1, \ldots, P_{j-1}]$ is defined, and
 (ii) the lift $\gamma_{j-1}(P_j)$ is a closed p-submanifold of $[M : P_1, \ldots, P_{j-1}]$.

4.2. **Disjoint submanifolds.** A particularly simple instance of the iterated blow-up is when the corresponding manifolds are disjoint. Nevertheless, this is an important case that will be discussed in this subsection. This will also alow us to introduce and prove the first factorization lemma, Lemma 4.7. It also allows us to deal with the case of a closed p-submanifold that has connected components of different dimensions, which, we recall, is allowed. In particular, blowing-up with respect to such a manifold amounts, as we will see, to blowing up successively with respect to each component.

We need first to discuss the gluing of open subsets. Let us assume that we have two manifolds with corners M_1 and M_2 and that $U_i \subset M_i$ are open subsets ($i = 1, 2$). Let us also assume that we are given a diffeomorphism $\phi : U_1 \to U_2$. Then we define

$$M_1 \cup_\phi M_2 := (M_1 \sqcup M_2)/\{x \equiv \phi(x) \mid x \in U_1\},$$

(23)
$$M_1 \cup_{\mathrm{id}} M_2 =: M_1 \cup_{U_1} M_2, \quad \text{if } U_1 = U_2 \text{ and } \phi \text{ is the identity map id}.$$

If ϕ is the identity, we shall call $M_1 \cup_{U_1} M_2$ the *union of M_1 and M_2 along $U_1 = U_2$*. Under favorable circumstances (but not always), $M_1 \cup_\phi M_2$ is also a manifold with corners.

We have the following simple lemma.

Lemma 4.4. *Let M be a manifold with corners (and hence Hausdorff) and $M_i \subset M$, $i = 1, 2$, be open subsets with $U := M_1 \cap M_2$ and $M_1 \cup M_2 = M$. Then there exists a unique structure of a manifold with corners on $M_1 \cup_U M_2$ that induces the given smooth structures on M_i, and hence we have a canonical diffeomorphism $M_1 \cup_U M_2 \simeq M$.*

Proof. Let $\mathcal{A}_i$ be an atlas for M_i. Then their union $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas for M. It is also an atlas for any manifold with corners structure on $M_1 \cup_U M_2$ that induces the given one on each M_i. Hence the desired manifold with corners structure on $M_1 \cup_U M_2$ is given by the union $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$. $\square$

This lemma allows us to "commute" the blow-ups with respect to disjoint closed p-submanifolds. We thus have the following simple result, see for example [1, 48, 56].

Lemma 4.5. *Let P and Q be closed p-submanifolds of M and $\beta_{M,Q} : [M : Q] \to M$ be the blow-down map. Assume that $P \cap Q = \emptyset$. Then $\beta_{M,Q}^*(P) := \beta_{M,Q}^{-1}(P) = P$ and the iterated blow-up $[M : Q, P] := \big[[M : Q] : P\big]$ is defined and diffeomorphic to $([M : Q] \smallsetminus P) \sqcup_{M \smallsetminus (P \cup Q)} ([M : P] \smallsetminus Q)$, the union of $[M : Q] \smallsetminus P$ and $[M : P] \smallsetminus Q$ along $M \smallsetminus (P \cup Q)$, a common open subset. In particular, by symmetry,*

$$\big[[M : P] : Q\big] \;=\; [M : P \cup Q] \;=\; \big[[M : Q] : P\big],$$

with the same smooth structure.

The idea of the proof is to use the fact that the construction of the blow-up is local in nature.

Definition 4.6. Suppose $f_i : X \to Y_i$, $i = 1, \ldots, N$, are continuous maps. We say $(f_1, \ldots, f_N) : X \to \prod_{i=1}^N Y_i$, $x \mapsto (f_1(x), \ldots, f_N(x))$ is proper in each component if each f_i is proper.

We shall need the following "factorization" lemma. The existence of the map $\zeta_{M,Q,P}$ of follows from Lemma 4.5 or from a more general result in [48].

Lemma 4.7 (The first factorization Lemma)**.** *Let us assume that P and Q are closed, disjoint p-submanifolds of M. Then there exists a unique, smooth, natural map*

$$\zeta_{M,Q,P} : \big[[M : Q] : P\big] \;\to\; [M : P]$$

that restricts to the identity on $M \setminus (P \cup Q)$. Moreover, the product map

$$\mathcal{B}_{M,Q,P} := (\zeta_{M,Q,P}, \beta_{[M:Q],P}) : \big[[M:Q]:P\big] \to [M:P] \times [M:Q]$$

is proper in each component. Its image is a weak submanifold in the sense of Definition 2.10 and $\mathcal{B}_{M,Q,P}$ is a diffeomorphism onto its image.

Again, our main focus lies on the case $\dim(P) < \dim(M)$ and $\dim(Q) < \dim(M)$. The statements, however remain (trivially) true if one of these dimensions is equal to $\dim(M)$ (equivalently, if a connected component of M is contained in P or Q). Then this component is removed both from $\big[[M:Q]:P\big]$ and from $[M:P]$ or $[M:Q]$.

Proof. Lemma 4.5 states that $\big[[M:Q]:P\big] = [M:P \cup Q] = \big[[M:P]:Q\big]$. This gives $\zeta_{M,Q,P} = \beta_{[M:P],Q}$. In particular, $\zeta_{M,Q,P}$ is proper, by Corollary 3.8. The map $\beta_{[M:Q],P}$ is proper by Corollary 3.8. As P and Q are disjoint, at each point, at least one component of $\mathcal{B}_{M,Q,P} = (\zeta_{M,Q,P}, \beta_{[M:Q],P})$ is a local diffeomorphism. Thus $\mathcal{B}_{M,Q,P}$ is an immersion. As it is injective and proper, it is a homeomorphism onto its image. Proposition 2.13 implies that the image is thus a weak submanifold and that $\mathcal{B}_{M,Q,P}$ is a diffeomorphism onto its image. $\qquad\square$

By iterating the above lemma, we obtain the following consequence.

Corollary 4.8. *Let $\mathcal{P} := (\emptyset, P_1, P_2, \ldots, P_k)$ be a family of closed,* disjoint *p-submanifolds of a manifold with corners M. Then we have canonical diffeomorphisms inducing the identity on $M \setminus \bigcup \mathcal{P} := M \setminus \bigcup_{j=1}^k P_j$ between the usual blow-ups and the graph blow-up (Definitions 3.1 and 4.17):*

$$[[\ldots[[M:P_1]:P_2]: \ldots :P_{k-1}]:P_k] \simeq [M:\bigcup_{j=1}^k P_j] \simeq \{M:\mathcal{P}\}.$$

Proof. This follows by induction from Lemmas 4.5 and 4.7 since P_j identifies naturally with a p-submanifold of $[[\ldots[[M:P_1]:P_2]: \ldots :P_{j-2}]:P_{j-1}]$. $\qquad\square$

4.3. **Clean semilattices.** We now investigate the iterated blow-up $[M:(P_i)_{i=1}^k]$ of a manifold with corners M with respect to a suitable cleanly intersecting finite *totally ordered* family of closed p-submanifolds of M. (If we arrange the elements of this family according to the total order, then we obtain a k-tuple.)

Definition 4.9. Let $\mathcal{F}$ be a locally finite (unordered) set of closed p-submanifolds of M. We shall say that $\mathcal{F}$ is a *cleanly intersecting family* if any $X_1, X_2, \ldots, X_j \in \mathcal{F}$ have a clean intersection (Definition 3.11).

Recall that a *meet semilattice* (or, simply, *semilattice* in what follows) is a partially ordered set $\mathcal{L}$ such that, for every two $x, y \in \mathcal{L}$, there is a greatest common lower bound $x \cap y \in \mathcal{L}$ for x and y. We shall consider only semilattices of subsets of a given set where the order is given by $\subset$ and where $x \cap y$ is the usual intersection of sets. We can now introduce the semilattices we are interested in. We let 2^M denote the set of all subsets of M. Thus all our semilattices will be subsets of M that are stable for intersection. The order given by inclusion will not play a role and will thus be ignored.

Definition 4.10. A semilattice $\mathcal{S} \subset 2^M$ will be called *clean* if it is a cleanly intersecting locally finite set of closed p-submanifolds of M.

We stress that this definition requires that all elements of a clean semilattice are closed p-submanifolds of M. It is easier to deal with semilattices since a semilattice of closed p-submanifolds is clean if, and only if, any two members of the semilattice intersect cleanly. For simplicity of the notation, we shall consider only semilattices $\mathcal{S} \subset 2^M$ (the set of subsets of M) with $\emptyset \in \mathcal{S}$. This changes nothing in our results, by Remark 4.2, but avoids us treating separately the cases $\emptyset \in \mathcal{S}$ and $\emptyset \notin \mathcal{S}$ in proofs.

The concept of a clean semilattice introduced here is very closely related to that of a weakly transversal family considered in [1, Definition 2.7], except that in that paper, the authors considered only p-submanifolds that were *not* contained in the boundary. Similar concepts were also considered in [48, Theorem 3.2] and in [71]. The case considered in [71, Sec. 2] was that when all p-submanifolds with respect to which we blow-up *are contained in the boundary.*

The following result was proved in special cases in [1] and in [71, Lemma 2.7] with similar proofs. (See also [48].) In [56], Melrose proved that a the lift of a normal family remains a normal family if we do the blow-up by an element of the family. Lemma 5.11.2 of [56] also treats the lift of a family under the blow-up.

Proposition 4.11. *Let $\emptyset \in \mathcal{S} \subset 2^M$ be a clean semilattice. (Thus $\mathcal{S}$ consists of closed p-submanifolds of M.) Let P be a minimal element of $\mathcal{S} \smallsetminus \{\emptyset\}$. Then, for all $Q \in \mathcal{S}$, $Q' := [Q : P \cap Q]$ is a closed p-submanifold of $[M : P]$ and (after we remove the repetitions)*

$$\mathcal{S}' := \Big\{ Q' = [Q : P \cap Q] \mid Q \in \mathcal{S} \Big\}$$

is a clean semilattice of $[M : P]$ with $\emptyset = \emptyset' = P' \in \mathcal{S}'$ and, hence, with fewer elements than $\mathcal{S}$.

Proof. The first part of tis result was proved in slightly less generality in [1, Theorem 2.8] (assuming that the p-submanifolds are *not* contained in the boundary). The proof extends right away to the current setting using Lemma 3.14. See also [48].

For the last part of the result, recall that the minimality of P and the semilattice property of $\mathcal{S}$ imply that, for any $Q \in \mathcal{S}$, we have either $P \subset Q$ or $P \cap Q = \emptyset$. In the first case, we have $Q' := [Q : P \cap Q] = [Q : P]$ and in the second case we have $Q' := [Q : P \cap Q] = Q$. Thus

$$\mathcal{S}' := \{ [Q : P] \mid P \subset Q \in \mathcal{S} \} \cup \{ Q \mid Q \in \mathcal{S}, Q \cap P = \emptyset \}.$$

Let us also notice that $P' := [P : P \cap P] = \emptyset = [\emptyset : \emptyset \cap P] = \emptyset'$. Therefore, $\mathcal{S}'$ has fewer elements than $\mathcal{S}$. $\qquad\square$

We are ready now to describe what are the k-tuples $\mathcal{P}$ with respect to which we will consider the blow-up $[M : \mathcal{P}]$: they are k-tuples coming from a total order on a set of closed p-submanifolds of M (denoted by abuse of notation still $\mathcal{P}$) that contains $\emptyset$ and is stable by intersection (thus, a semilattice). In addition to this, our semilattice $\mathcal{P}$ needs to be clean and the total order on $\mathcal{P}$ needs to be "admissible," a concept that we define next.

Definition 4.12. Let $\mathcal{S} \ni \emptyset$ be a *finite,* clean semilattice of closed p-submanifolds of M. We define by induction an *admissible order on $\mathcal{S}$* to be a total order $\mathcal{S} = (P_0 = \emptyset, P_1, \ldots, P_n)$ on $\mathcal{S}$ (thus different, in general, from the inclusion order) such that P_1 is minimal for inclusion in $\mathcal{S} \smallsetminus \{\emptyset\}$ and the resulting ordering on $\mathcal{S}'$ is also admissible. (Recall that $\mathcal{S}'$ has fewer elements than $\mathcal{S}$). If $\mathcal{S}$ has only one element, then its order is admissible. If $\mathcal{S}$ has two elements, then the order is admissible if $\emptyset$ is minimal.

We have the following important remark.

Remark 4.13. Proposition 4.11 allows us to define the iterated blow-up $[M : \mathcal{S}]$ with respect to an admissible order on $\mathcal{S}$ by induction by

$$[M : \mathcal{S}] := [[M : P_1] : \mathcal{S}'],$$

where we used the notation of that proposition. This was the approach in [1] for the definition of $[M : \mathcal{S}]$. A very similar method was used by Kottke in [48], but using size orders (recalled in Remark 4.2). For our version of the iterated blow-up, the concept of size order, while very close to that of an admissible order, does not work, since $\mathcal{S}'$ may not be size ordered anymore, even if $\mathcal{S}$ was size ordered. Nevertheless, we can always choose an admissible order on the semilattice $\mathcal{S}$ of p-submanifolds of M that is also a size order. (Of course, in view of Remark 4.2, we can assume that the order on any k-tuple with respect to which we blow-up is a size-order, but that is a weaker statement.)

4.4. **The pair blow-up lemma.** We now perform some essential calculations in local coordinates that will be needed for our main result. Recall from Equation (12) that

$$\mathbb{S}^n_k := \mathbb{S}^n \cap \mathbb{R}^{n+1}_k,$$

where $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$, as always. For $\psi \in \mathbb{S}^{n'+1}_{k'+1} := \mathbb{S}^{n'+1} \cap \mathbb{R}^{n'+2}_{k'+1}$, we shall write $\psi =: (\psi_1, \tilde{\psi})$, with $\psi_1 \in [0,1]$ and $\tilde{\psi} \in \mathbb{R}^{n'+1}_{k'}$, and we define the map

(24)
$$\Upsilon : \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1} \to \mathbb{S}^{n,n'}_{k,k'} := \mathbb{S}^{n+n'} \cap \left(\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} \right)$$
$$(\phi, \psi) \mapsto (\psi_1 \phi, \tilde{\psi}).$$

We embed the sphere orthant $\{0\} \times \mathbb{S}^{n'}_{k'} = \{0_{\mathbb{R}^n}\} \times \mathbb{S}^{n'}_{k'} \subset \mathbb{R}^{n+n'+1}$ into $\mathbb{R}^{n+n'+1}$ by mapping the sphere orthant to the *last* components of $\mathbb{R}^{n+n'+1}$. Of course, we have an isomorphism

$$\mathbb{S}^{n,n'}_{k,k'} = \mathbb{S}^{n+n'} \cap \left(\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} \right) \cong \mathbb{S}^{n+n'}_{k+k'} = \mathbb{S}^{n+n'} \cap \mathbb{R}^{n+n'+1}_{k+k'}$$

given by the canonical permutation of coordinates diffeomorphism of Equation (14).

We recall Proposition 5.8.1 of [56] and we give the proof to fix the notation.

Lemma 4.14. *Let again* $\mathbb{S}^{n,n'}_{k,k'} := \mathbb{S}^{n+n'} \cap \left(\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} \right) \cong \mathbb{S}^{n+n'}_{k+k'}$ *and let the map* $\Upsilon : \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1} \to \mathbb{S}^{n,n'}_{k,k'}$ *be as in Equation* (24)*. If we define*

$$\Psi : \mathbb{S}^{n,n'}_{k,k'} \smallsetminus \left(\{0\} \times \mathbb{S}^{n'}_{k'} \right) \to \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}, \quad \Psi(\eta, \mu) = \left(\frac{\eta}{|\eta|}, \left(|\eta|, \mu \right) \right),$$

then $\Upsilon \circ \Psi$ *is the inclusion* $\mathbb{S}^{n,n'}_{k,k'} \smallsetminus \left(\{0\} \times \mathbb{S}^{n'}_{k'} \right) \subset \mathbb{S}^{n,n'}_{k,k'}$ *and* Ψ *extends to a diffeomorphism*

$$\widetilde{\Psi} : [\mathbb{S}^{n,n'}_{k,k'} : \{0\} \times \mathbb{S}^{n'}_{k'}] \xrightarrow{\sim} \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}$$

such that $\beta_{\mathbb{S}^{n,n'}_{k,k'}, \{0\} \times \mathbb{S}^{n'}_{k'}} = \Upsilon \circ \widetilde{\Psi}$.

If we write by abuse of notation $\mathbb{S}^{n'}_{k'}$ for the image of $\{0\} \times \mathbb{S}^{n'}_{k'}$ in $\mathbb{S}^{n+n'}_{k+k'}$ under the permutation of coordinates described above, then we obtain a diffeomorphism

$$\widetilde{\Psi} : [\mathbb{S}^{n+n'}_{k+k'} : \mathbb{S}^{n'}_{k'}] \xrightarrow{\sim} \mathbb{S}^{n-1}_k \times \mathbb{S}^{n'+1}_{k'+1}.$$

Proof. Let

$$\beta := \beta_{\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}, \{0\} \times \mathbb{R}^{n'+1}_{k'}} : [\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'} : \{0\} \times \mathbb{R}^{n'+1}_{k'}] \to \mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}.$$

denote the blow-down map. Also, recall that the lifting $\beta^*(\mathbb{S}_{k,k'}^{n,n'})$ is defined as the closure of $\beta^{-1}(\mathbb{S}_{k,k'}^{n,n'} \smallsetminus \{0\} \times \mathbb{R}_{k'}^{n'+1})$ in $[\mathbb{R}_k^n \times \mathbb{R}_{k'}^{n'+1} : \{0\} \times \mathbb{R}_{k'}^{n'+1}]$. Since

$$\mathbb{S}_{k,k'}^{n,n'} \cap \left(\{0\} \times \mathbb{R}_{k'}^{n'+1}\right) \;=\; \{0\} \times \mathbb{S}_{k'}^{n'},$$

Proposition 3.14 gives a diffeomorphism

$$\Phi : [\mathbb{S}_{k,k'}^{n,n'} : \{0\} \times \mathbb{S}_{k'}^{n'}] \;\xrightarrow{\sim}\; \beta^*(\mathbb{S}_{k,k'}^{n,n'}),$$

uniquely determined by the condition that it is the inclusion on $\mathbb{S}_{k,k'}^{n,n'} \smallsetminus \{0\} \times \mathbb{S}_{k'}^{n'}$. (That is, the blow-up of $\mathbb{S}_{k,k'}^{n,n'}$ along $\{0\} \times \mathbb{S}_{k'}^{n'}$ is diffeomorphic to the lifting $\beta^*(\mathbb{S}_{k,k'}^{n,n'})$ of $\mathbb{S}_{k,k'}^{n,n'}$ to $[\mathbb{R}_k^n \times \mathbb{R}_{k'}^{n'+1} : \{0\} \times \mathbb{R}_{k'}^{n'+1}]$ via the blow-down map $\beta := \beta_{\mathbb{R}_k^n \times \mathbb{R}_{k'}^{n'+1}, \{0\} \times \mathbb{R}_{k'}^{n'+1}}$.)

To identify more explicitly the space $\beta^*(\mathbb{S}_{k,k'}^{n,n'})$, it is convenient to use the diffeomorphism $\kappa : \mathbb{S}_k^{n-1} \times [0, +\infty) \times \mathbb{R}_{k'}^{n'+1} \to [\mathbb{R}_k^n \times \mathbb{R}_{k'}^{n'+1} : \{0\} \times \mathbb{R}_{k'}^{n'+1}]$ of Equation (19) with the order of its arguments reversed. To start with, the blow-down map $\beta := \beta_{\mathbb{R}_k^n \times \mathbb{R}_{k'}^{n'+1}, \{0\} \times \mathbb{R}_{k'}^{n'+1}}$ is such that $\beta_1 := \beta \circ \kappa$ satisfies

$$\beta_1 := \beta \circ \kappa : \mathbb{S}_k^{n-1} \times [0, +\infty) \times \mathbb{R}_{k'}^{n'+1} \to \mathbb{R}_k^n \times \mathbb{R}_{k'}^{n'+1},$$
$$\beta_1(z, r, x) \;=\; (rz, x).$$

We have that $(z, r, x) \in \beta_1^{-1}(\mathbb{S}_{k,k'}^{n,n'} \smallsetminus (\{0\} \times \mathbb{R}_{k'}^{n'+1}))$ if, and only if $\|\beta_1(z, r, x)\| = 1$ and $r > 0$. Assume that $\|\beta_1(z, r, x)\| = 1$ and $r > 0$. Then $\|rz\|^2 + \|x\|^2 = 1$. Note that $z \in \mathbb{S}_k^{n-1}$, and hence $r^2 + \|x\|^2 = 1$. This leads to $(r, x) \in \mathbb{S}_{k'+1}^{n'+1} \subset \mathbb{R}_{k'+1}^{n'+2} = [0, \infty) \times \mathbb{R}_{k'}^{n'+1}$. We thus have

$$\beta_1^{-1}\left(\mathbb{S}_{k,k'}^{n,n'} \smallsetminus (\{0\} \times \mathbb{R}_{k'}^{n'+1})\right) \;=\; (\mathbb{S}_k^{n-1} \times \mathbb{S}_{k'+1}^{n'+1}) \smallsetminus (\{0\} \times \mathbb{R}_{k'}^{n'+1}).$$

The closure of this set is $\mathbb{S}_k^{n-1} \times \mathbb{S}_{k'+1}^{n'+1}$, and hence we obtain a diffeomorphism $\Phi_1 := \kappa^{-1} \circ \Phi : [\mathbb{S}_{k,k'}^{n,n'} : \{0\} \times \mathbb{S}_{k'}^{n'}] \xrightarrow{\sim} \mathbb{S}_k^{n-1} \times \mathbb{S}_{k'+1}^{n'+1}$. That $\Upsilon \circ \Psi$ is the inclusion follows from the defining formulas. The relation $\beta_{\mathbb{S}_{k,k'}^{n,n'}, \{0\} \times \mathbb{S}_{k'}^{n'}} = \Upsilon \circ \widetilde{\Psi}$ follows from the fact that they are both continuous and they coincide on the dense, open subset $\mathbb{S}_{k,k'}^{n,n'} \smallsetminus (\{0\} \times \mathbb{S}_{k'}^{n'})$. This shows that $\Phi_1 = \Psi$ on $\mathbb{S}_{k,k'}^{n,n'} \smallsetminus (\{0\} \times \mathbb{S}_{k'}^{n'})$ and hence $\tilde{\Psi} := \Phi_1$ is the desired extension. $\qquad\square$

We now treat another basic case when the blow-up with respect to a clean semilattice consisting of three manifolds is defined, namely the simplest case when we blow up with respect to $\emptyset$ and two closed p-submanifolds P and Q with $\emptyset \neq Q \subset P$. This gives the "second factorization lemma." (The other basic case is that two disjoint p-submanifolds, which was already treated in Lemma 4.7.) Again, the existence of the map $\zeta_{M,Q,P}$ was proved in [48], but for a slightly different version of the iterated blow-up.

Lemma 4.15 (The second factorization lemma). *Let us assume that Q is a closed p-submanifold of P and that P is a closed p-submanifold of M (and hence Q is also a p-submanifold of M). Then $[P : Q]$ is canonically diffeomorphic to a p-submanifold of $[M : Q]$ and there exists a unique, smooth, natural map*

$$\zeta_{M,Q,P} : [M : Q, P] \simeq \big[[M : Q] : [P : Q]\big] \;\to\; [M : P]$$

that restricts to the identity on $M \smallsetminus P$. Moreover, the product map

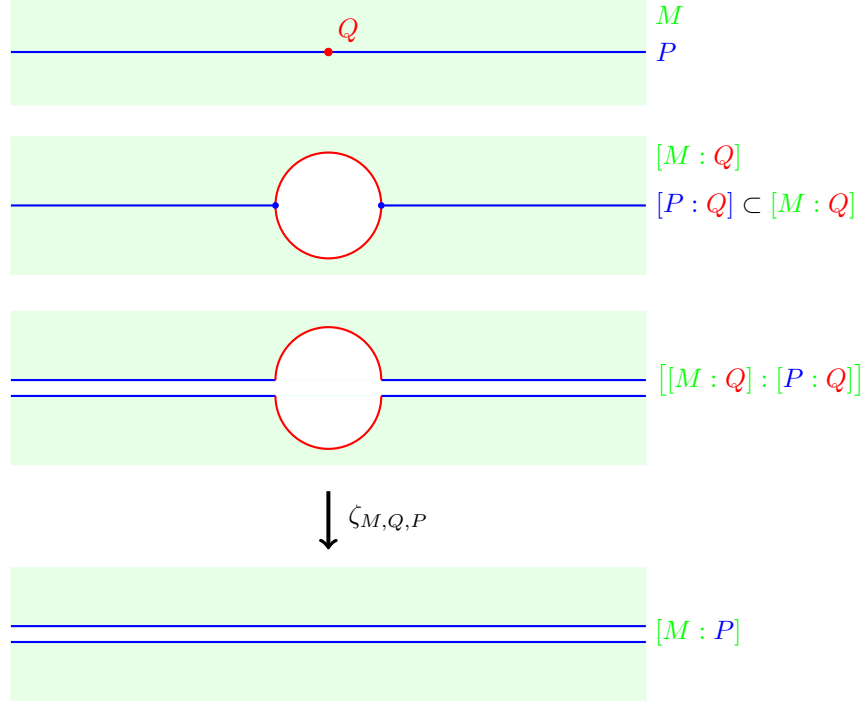$$\mathcal{B}_{M,Q,P} \;:=\; (\zeta_{M,Q,P}, \beta_{[M:Q],[P:Q]}) : [M : Q, P] \to [M : P] \times [M : Q]$$

FIGURE 2. The blow-ups $[M:Q]$, $\big[[M:Q]:[P:Q]\big]$, and $[M:P]$

is proper in each component. The image of $\mathcal{B}_{M,Q,P}$ is a weak submanifold in the sense of Definition 2.10, and $\mathcal{B}_{M,Q,P}$ is a diffeomorphism onto its image.

See Figure 4.4 for a local picture of these blow-ups in the example $M = \mathbb{R}^2$, $P = \mathbb{R} \times \{0\}$, $Q = \{0\}$.

Proof. The fact that $[P:Q]$ is (diffeomorphic to) a closed p-submanifold of $[M:Q]$ follows from 3.14. The uniqueness of the map $\zeta_{M,Q,P}$ follows from the fact that it is the identity on the dense subset $M \smallsetminus (P \cup Q)$. The statement is local, so, in view of Lemma 3.9, we can assume that $Q = \{0\}$. That is, we can assume that

(25)
$$\begin{cases} M & := & \mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p}\,, \\ P & := & \{0\} \times \mathbb{R}^p_{k_p}\,, \\ Q & := & \{0\}\,. \end{cases}$$

We have

$$\begin{aligned} [M:P] &= [\mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} : \{0\} \times \mathbb{R}^p_{k_p}] \\ &= [\mathbb{R}^m_{k_m} : \{0\}] \times \mathbb{R}^p_{k_p} \\ &\simeq \mathbb{S}^{m-1}_{k_m} \times [0,\infty) \times \mathbb{R}^p_{k_p} \\ &= \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1}\,. \end{aligned}$$

Its blow-down map is $\beta_{M,P}(x,t,y) = (tx,y)$.

On the other hand, we have (using the notation of Lemma 4.14):

$$[M:Q] \; = \; [\mathbb{R}^{m+p}_{k_m+k_p} : \{0\}] \; = \; \mathbb{S}^{m,p-1}_{k_m,\,k_p} \times [0,\infty).$$

Its blow-down map is $\beta_{M,Q}(x,t) = tx$. Lemma 3.14 gives that the lift of P to $[M:Q]$ is $P' := [P:Q] = \{0\} \times \mathbb{S}^{p-1}_{k_p} \times [0,\infty)$. Lemmas 3.9 and 4.14 (in this order) then give canonical diffeomorphisms

$$
\begin{aligned}
\big[[M:Q]:P'\big] \; &\simeq \; [\mathbb{S}^{m,p-1}_{k_m,\,k_p} \times [0,\infty) : \mathbb{S}^{p-1}_{k_p} \times [0,\infty)] \\
&= \; [\mathbb{S}^{m,p-1}_{k_m,\,k_p} : \mathbb{S}^{p-1}_{k_p}] \times [0,\infty) \\
&\simeq \; \mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^{p}_{k_p+1} \times [0,\infty)\,.
\end{aligned}
$$

The blow-down map $\beta_{[[M:Q]:P]} : \big[[M:Q]:P'\big] \to [M:Q]$ is given, up to canonical diffeomorphisms, by the map $\Upsilon \times \mathrm{id}$, where Υ is as defined in Equation (24). Hence $\Upsilon \times \mathrm{id}(\phi,\psi,t) = (\psi_1\phi,\tilde{\psi},t)$.

The desired map $\zeta_{M,Q,P}$ is then obtained from the blow-down map $\mathbb{S}^{p}_{k_p+1} \times [0,\infty) \to \mathbb{R}^{p+1}_{k_p+1} = [0,\infty) \times \mathbb{R}^{p}_{k_p}$, that is $\zeta_{M,Q,P}(x,y,t) = (x,ty)$. In particular, it is proper. It remains to check that this map is the identity on $M \smallsetminus P$. Since we used, for $x \in M \smallsetminus P$, the identifications $x = \beta_{M,Q}(x) = \beta_{M,P}(x) = \beta_{[M:Q],[P:Q]}(x)$, it is enough to check

$$(26) \qquad\qquad \beta_{M,P} \circ \zeta_{M,Q,P} = \beta_{M,Q} \circ \beta_{[M:Q],[P:Q]}$$

on $M \smallsetminus P$. As this calculation is local, we can again assume (25) and the concrete presentations of $\beta_{M,Q}$, $\beta_{M,P}(x)$ and $\beta_{[M:Q],[P:Q]}$ described above, (26) turns into

$$(27) \qquad\qquad \beta_{M,P} \circ \zeta_{M,Q,P} = \beta_{M,Q} \circ (\Upsilon \times \mathrm{id})$$

on $\mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^{p}_{k_p+1} \times [0,\infty)$. Indeed, for $x \in \mathbb{S}^{m-1}_{k_m}$, $y = (y_1,\tilde{y}) \in \mathbb{S}^{p}_{k_p+1} \subset \mathbb{R}^{p+1}_{k_p+1} = \mathbb{R}^1_1 \times \mathbb{R}^{p}_{k_p}$, $t \in [0,\infty) = \mathbb{R}^1_1$, we have

$$\beta_{M,P} \circ \zeta_{M,Q,P}(x,y,t) \; = \; \beta_{M,P}(x,ty) \; = \; \beta_{M,P}(x,ty) \; = \; (ty_1 x, t\tilde{y})\,.$$

Together with

$$\beta_{M,Q} \circ (\Upsilon \times \mathrm{id})(x,y,t) \; = \; \beta_{M,Q}(y_1 x, \tilde{y}, t) \; = \; (ty_1 x, t\tilde{y})\,,$$

this implies (27).

The map $\mathcal{B}$ is given in local coordinates by $\mathcal{B}(x,y,t) = \big(x, ty, (y_1 x, \tilde{y}), t\big)$ with differentiable left inverse $(x,z,(w_1,w_2),t) \mapsto (x,(\|w_1\|,w_2),t)$. Hence by Corollary 2.14 the image of $\mathcal{B}$ is a weak submanifold and $\mathcal{B}$ is a diffeomorphism onto its image. $\qquad\square$

Remark 4.16. Note that, in general, the image of the map $\mathcal{B}_{M,Q,P}$ introduced in the proof above is not a p-submanifold of $[M:P] \times [M:Q]$. Indeed, let us consider the case when M is the closed unit disk in $\mathbb{R}^2$, and let p and q be two disjoint points in the interior of M. Let $P := \{p\}$ and $Q := \{q\}$. We claim that the image N of $\mathcal{B} = \mathcal{B}_{M,Q,P}$ is *not* a p-submanifold of $M_1 := [M:P] \times [M:Q]$. Suppose N were a p-submanifold of M_1. As N is connected, the function $\mathrm{depth}_{M_1}(x) - \mathrm{depth}_N(x)$ is constant on N, see Remark 2.16. However, the map $\mathcal{B}$ sends the interior points of $M \smallsetminus \{p,q\}$ to the interior of $M_1 = [M:P] \times [M:Q]$, thus $\mathrm{depth}_{M_1}(x) - \mathrm{depth}_N(x) = 0 - 0 = 0$ for $x = \mathcal{B}(y)$ with y in the interior of $M \smallsetminus \{p,q\}$. On the other hand $\mathcal{B}$ maps the boundary $\partial M = \partial(M \smallsetminus \{p,q\})$ to the corner $\partial M \times \partial M$ of $[M:P] \times [M:Q]$, which has boundary depth 2 in $M_1 = [M:P] \times [M:Q]$. Thus, if $x = \mathcal{B}(y)$, with $y \in \partial M$, we obtain $\mathrm{depth}_{M_1}(x) - \mathrm{depth}_N(x) = 2 - 1 = 1$. Therefore, the function $\mathrm{depth}_{M_1}(x) - \mathrm{depth}_N(x)$ is not constant on N, and hence N is not a p-submanifold of $M_1 = [M:P] \times [M:Q]$.

A careful investigation [50] shows that the image $\mathcal{B}_{M,Q,P}([M:Q,P]$ of $[M:Q,P]$ in $[M:P] \times [M:Q]$ may fail to be a submanifold of $[M:P] \times [M:Q]$ in the sense of manifolds with corners, see Definition B.1. This fact justifies our introduction of the notion of a "weak submanifold." In particular, a weak submanifold is neither a b-submanifold nor a wib-submanifold, see Appendix B.2.

4.5. **The graph blow-up.** In this section we introduce the *graph blow-up*, which is a version of the blow-up with respect to a family of closed p-submanifolds that obviously does not depend on any order on that family. For our applications the most important case is the one of a compact manifold with corners M. In this case locally finiteness implies finiteness. Finiteness of the semilattice simplifies the presentation, thus we will assume this from now on; but let us mention that there are obvious extensions to locally finite clean semilattices in non-compact manifolds with corners.

Let M be a manifold with corners and $\mathcal{P} = (P_i)_{i \in I}$ be a finite family of closed p-submanifolds of M. Let $\delta : M \smallsetminus \bigcup \mathcal{P} \to \prod_{i \in I}[M : P_i]$ be the diagonal map $\delta(x) = (x, x, \ldots, x)$, as before. We write $\bigcup \mathcal{P} := \bigcup_{i \in I} P_i$. Then $M \smallsetminus \bigcup \mathcal{P}$ is an open subset of $[M : P_i]$, for each $i \in I$. Motivated by the results of [35, 61], we now introduce the following definition.

Definition 4.17. Let $\mathcal{P} = (P_i)_{i \in I}$ be a finite family of closed p-submanifolds of the manifold with corners M. Then the *graph blow-up* $\{M : \mathcal{P}\}$ of M along $\mathcal{P}$ is defined by

$$\{M : \mathcal{P}\} := \overline{\delta(M \smallsetminus \bigcup \mathcal{P})} = \overline{\{(x, x, \ldots, x) \mid x \in M \smallsetminus \bigcup \mathcal{P}\}} \subset \prod_{i \in I}[M : P_i].$$

Remark 4.18. The definition of the family $\mathcal{P} = (P_i)_{i \in I}$ allows for repetitions. That is, we may have $P_i = P_j$ for some $i, j \in I$, $i \neq j$. If we *remove the repetitions,* will obtain a graph blow-up that is canonically homeomorphic to the original one. Similarly, changing the index set I will also yield a canonically homeomorphic graph blow-up. In particular, if I is finite (which is the case for most of this paper), we can introduce a total order on $\mathcal{P}$. The graph blow-up corresponding to different orders on the finite set $\mathcal{P}$ will be, however, canonically homeomorphic. See also Remark 4.2.

We now show that $\{M : \mathcal{P}\}$ is a weak submanifold of a suitable manifold with corners provided that $\mathcal{P}$ is an admissible ordered clean semilattice.

Theorem 4.19. *Let $\mathcal{S} \ni \emptyset$ be a finite, clean semilattice of closed p-submanifolds of M with an admissible order (Definition 4.12), so that $[M : \mathcal{S}]$ is well-defined (Remark 4.13).*

(i) *For each $P \in \mathcal{S}$, there exists a unique smooth map $\phi_{\mathcal{S},P} : [M : \mathcal{S}] \to [M : P]$ that is the identity on $M \smallsetminus \bigcup_{P \in \mathcal{S}} P$. These maps are such that the induced map*

$$\mathcal{B}_{\mathcal{S}} := (\phi_{\mathcal{S},P_0}, \ldots, \phi_{\mathcal{S},P_k}) : [M : \mathcal{S}] \to \prod_{j=0}^{k}[M : P_j]$$

is an injective immersion and proper in each component.

(ii) *The image of $\mathcal{B}_{\mathcal{S}}$ is $\{M : \mathcal{S}\}$. Hence, $\{M : \mathcal{S}\}$ is a weak submanifold of the product $\prod_{j=0}^{k}[M : P_j]$ in the sense of Definition 2.10 and $\mathcal{B}_{\mathcal{S}}$ is a diffeomorphism*

$$\mathcal{B}_{\mathcal{S}} : [M : \mathcal{S}] \xrightarrow{\sim} \{M : \mathcal{S}\}.$$

Proof. We shall prove (i) and (ii) together by induction on the number $k + 1$ of elements of $\mathcal{S}$. Recall that the iterated blow-up $[M : \mathcal{S}]$ is defined since $\mathcal{S}$ is endowed with an

admissible order. (See also Remark 4.13.) In the case $k = 0$, there is nothing to prove, since $\mathcal{S} = \{\emptyset\}$ then.

Case $k = 1$: If $\mathcal{S}$ has $1 + 1 = 2$ elements, we have $\mathcal{S} = (\emptyset, P)$ and $\mathcal{B}_\mathcal{S} = (\beta_{M,P}, \mathrm{id}_{[M:P]})$ so the claim is trivially satisfied, since the blow-down map is proper (Corollary 3.8).

Case $k = 2$: If $\mathcal{S}$ has $2 + 1 = 3$ elements, we have $\mathcal{S} = \{\emptyset, Q, P\}$ and either $Q \cap P = \emptyset$ or $Q \subset P$ (the case $P \subset Q$ would not yield an admissible order on $\mathcal{S}$).

1) In the *first* subcase, that is, if $Q \cap P = \emptyset$, the result was already proved in the first factorization lemma, Lemma 4.7, with $\mathcal{B}_\mathcal{S} = (\beta_{M,P\cup Q}, \beta_{[M:Q],P}, \beta_{[M:P],Q})$, that is, all the components of $\mathcal{B}_\mathcal{S}$ are given by blow-down maps. The diffeomorphism property for $\mathcal{B}_\mathcal{S}$ comes from the fact that its restriction to $[M \smallsetminus Q : P]$ and $[M \smallsetminus P : Q]$ has a component equal to the identity, so it is a local diffeomorphism onto its image, which is at the same time injective and proper, thus having a continuous inverse.

2) Similarly, in the *second* subcase, that is, if $Q \subset P$, the result was already proved in the second factorization lemma, Lemma 4.15, with

$$\mathcal{B}_\mathcal{S} := \left(\beta_{M,Q} \circ \beta_{[M:Q],[P:Q]}, \beta_{[M:Q],[P:Q]}, \zeta_{M,Q,P}\right),$$

that is, we have, $\phi_{\mathcal{S},\emptyset} = \beta_{M,Q} \circ \beta_{[M:Q],[P:Q]}$, $\phi_{\mathcal{S},Q} := \beta_{[M:Q],[P:Q]}$, $\phi_{\mathcal{S},P} := \zeta_{M,Q,P}$. In particular, the fact that $\mathcal{B}_{M,Q,P} = (\beta_{[M:Q],[P:Q]}, \zeta_{M,Q,P})$ is a diffeomorphism onto its image implies the same statement for $\mathcal{B}_\mathcal{S}$.

Case $k \geq 3$: Let us now proceed with the induction step from $k - 1$ to k, that is, let us assume that $\mathcal{S}$ has $k + 1$ elements, $P_0 = \emptyset, P_1, \ldots, P_k$, arranged in the given, admissible order. Let $P' := [P : P \cap P_1]$. Thus we have $P' = [P : P_1]$, if $P_1 \subset P$, and $P' = P$, if $P_1 \cap P = \emptyset$. We shall use the notation of Proposition 4.11 with $P := P_1$, in particular, $Q' := [Q : Q \cap P_1]$. The semilattice $\mathcal{S}' = \left(P'_j := [P_j : P_j \cap P_1]\right)_{j=1,\ldots k}$ of Proposition 4.11 is then clean and with an admissible order. As we have remarked already, $P'_1 := [P_1 : P_1 \cap P_1] = \emptyset = \emptyset'$, and hence $\mathcal{S}'$ has at most k elements. By the induction hypothesis, the map $\mathcal{B}_{\mathcal{S}'}$ is a diffeomorphism onto its image. The same property is shared by the maps

$$\mathcal{B}_{M,P_1,P_j} : \left[[M : P_1] : [P_j : P_1]\right] \to [M : P_1] \times [M : P_j]$$

of the Lemmata 4.7 and 4.15, since either $P_1 \cap P_j = \emptyset$ or $P_1 \subset P_j$, since we have assumed that the order on $\mathcal{S}$ is admissible. Let $\Phi := \mathrm{id} \times \prod_{j=2}^k \mathcal{B}_{M,P_1,P_j}$ and consider the composition

$$(28) \quad [M : \mathcal{S}] := \left[[M : P_1] : \mathcal{S}'\right] \xrightarrow{\mathcal{B}_{\mathcal{S}'}} \prod_{j=1}^k \left[[M : P_1] : [P_j : P_1]\right]$$

$$\xrightarrow{\Phi} [M : P_1] \times \prod_{j=2}^k \left([M : P_1] \times [M : P_j]\right).$$

The two maps of the composition are both injective immersions, and hence their composition is again an injective immersion. The desired map $\phi_{\mathcal{S},P_j}$ is the projection onto the P_j-component. The projection of the composite map onto any of the factors is the identity on $M \smallsetminus \left(\bigcup_{j=1}^k P_j\right)$. Note that all components with factors of the form $[M : P_1]$ (which are repeated), yield the same projection, again because this projection is the identity map on $M \smallsetminus \left(\bigcup_{j=1}^k P_j\right)$. By removing these repetitions, and by adding the iterated blow-down

map $[M : \mathcal{S}] \to M$ we obtain the desired map $\mathcal{B}_\mathcal{S}$, which is consequently also an injective immersion. The map $\mathcal{B}_\mathcal{S}$, is proper in each component, and thus proper. It follows from Corollary A.2 that $\mathcal{B}_\mathcal{S}$ is a homeomorphism to its image $N := \mathcal{B}_\mathcal{S}([M : \mathcal{S}])$. With Proposition 2.13 we see that N is a weak submanifold of $\prod_{j=0}^{k}[M : P_j]$, and that $\mathcal{B}_\mathcal{S}$ is a diffeomorphism onto N.

It remains to argue that N coincides with

$$\{M : \mathcal{S}\} := \overline{\mathcal{B}_\mathcal{S}\Big(M \smallsetminus \bigcup_{j=1}^{k} P_j\Big)}.$$

For any $x \in [M : \mathcal{S}]$, there is a sequence (x_i) in $M \smallsetminus \Big(\bigcup_{j=1}^{k} P_j\Big)$ converging to x in $[M : \mathcal{S}]$. Therefore

$$\mathcal{B}_\mathcal{S}\Big(M \smallsetminus (\bigcup_{j=1}^{k} P_j)\Big) \ni \mathcal{B}_\mathcal{S}(x_i) \to \mathcal{B}_\mathcal{S}(x),$$

and hence $\mathcal{B}_\mathcal{S}(x) \in \{M : \mathcal{S}\}$. It follows that $N \subset \{M : \mathcal{S}\}$.

Conversely, for $y \in \{M : \mathcal{S}\}$, there is a sequence $y_i = \mathcal{B}_\mathcal{S}(x_i)$ in $\mathcal{B}_\mathcal{S}\Big(M \smallsetminus \Big(\bigcup_{j=1}^{k} P_j\Big)\Big)$ converging to y in $\prod_{j=0}^{k}[M : P_j]$. Thus $\{y_i \mid i \in \mathbb{N}\} \cup \{y\}$ is compact, and by properness of $\mathcal{B}_\mathcal{S}$ the set

$$\big(\mathcal{B}_\mathcal{S}\big)^{-1}\left(\{y_i \mid i \in \mathbb{N}\} \cup \{y\}\right) = \{x_i \mid i \in \mathbb{N}\} \cup \big(\mathcal{B}_\mathcal{S}\big)^{-1}\left(\{y\}\right)$$

is compact as well. As a consequence a subsequence x_{i_k} has to converge to some $z \in [M : \mathcal{S}]$. We conclude that

$$N \ni \mathcal{B}_\mathcal{S}(z) = \lim_{l \to \infty} \mathcal{B}_\mathcal{S}(x_{i_l}) = \lim_{l \to \infty} y_{i_l} = y.$$

This yields $\{M : \mathcal{S}\} \subset N$. $\qquad\square$

Again, the image of the map $\mathcal{B}_\mathcal{S}$ is, in general, not a p-submanifold, see Remark 4.16. We obtain as a first application of our results the following actions of Lie groups.

Definition 4.20. If G is a Lie group acting smoothly on M and $\mathcal{P}$ is a finite set of closed p-submanifolds of M such that, for every $P \in \mathcal{P}$ and $g \in G$, we have $g(P) = Q$ for some $Q \in \mathcal{P}$, then we shall say that $\mathcal{P}$ *is a G-family of closed p-submanifolds of M*.

Proposition 3.10 yields right away the following corollary.

Theorem 4.21. *Let G be a Lie group acting smoothly on M and let $\mathcal{P}$ be a G-family of closed p-submanifolds of M (see Definition 4.20). Then G acts continuously on $\{M : \mathcal{P}\}$. If, moreover, $\mathcal{P}$ is a clean semilattice of closed p-submanifolds of M, then G commutes with the homeomorphism $\mathcal{B}_\mathcal{P}$ of Theorem 4.19 and it acts smoothly on $\{M : \mathcal{P}\} \simeq [M : \mathcal{P}]$.*

Proof. Let $\delta(x) = (x, \dots, x)$ be the diagonal embedding $\delta : M \smallsetminus \cup \mathcal{P} \to \prod_{P \in \mathcal{P}}[M : P]$ considered before. We have that each G acts smoothly on $M \smallsetminus \cup \mathcal{P}$ and on $\prod_{P \in \mathcal{P}}[M : P]$, with the action sending $[M : P]$ to $[M : g(P)]$, by Proposition 3.10. The action by homeomorphisms of G on $\{M : \mathcal{P}\}$ then follows since δ commutes with the action of G. The map $\mathcal{B}_\mathcal{P}$ is also clearly G-equivariant. The smoothness of the action of G on $\{M : \mathcal{P}\}$ then follows from Theorem 4.19 and the smoothness of the action of G on $\prod_{P \in \mathcal{P}}[M : P]$. $\qquad\square$

5. IDENTIFICATION OF THE GEORGESCU-VASY SPACE

We now apply the results of the previous sections to identify the spaces introduced by Georgescu and Vasy with the space $X_{\mathrm{GV}} := \overline{\delta_{\mathcal{F}}(X)}$ defined in the Introduction. In what follows, the role played by M in the previous sections will be played by the spherical compactification $\overline{Z}$ of a vector space Z, which we recall next.

5.1. **Spherical compactifications.** For any finite-dimensional real vector space Z, recall that $\mathbb{S}_Z$ denotes the set of vector directions in Z, that is, the set of (non-constant) open half-lines $\mathbb{R}_+v$, with $0 \neq v \in Z$ and $\mathbb{R}_+ := (0, \infty)$. The disjoint union

$$\overline{Z} := Z \sqcup \mathbb{S}_Z \tag{29}$$

is then called the *radial compactification* of Z. For example, if $Z = \mathbb{R}$, then $\overline{Z} := [-\infty, \infty]$ with the usual topology. The action of the group $\mathrm{GL}(Z)$ of linear automorphisms of Z extends, by definition, to an action on $\overline{Z}$. Similarly, if $Y \subset Z$, then $\overline{Y} \subset \overline{Z}$. In particular, $\overline{Z}$ is the union of all *closed lines* $\overline{\mathbb{R}v}$, $0 \neq v \in Z$, with closure taken in $\overline{Z}$.

As it is well known, $\overline{Z}$ carries a topology and a smooth structure, and our next goal is to recall their definitions, which will, in particular, turn $\overline{Z}$ into a smooth manifold with boundary. For notational purposes it is convenient consider the case $Z = \mathbb{R}^n$ first. We start by noticing that there is a bijection between the set of vector directions in $\mathbb{R}^{n+1}$ and its unit sphere $\mathbb{S}^n$. This allows us to regard $\mathbb{S}_1^n := \{(x_1, x') \in [0, \infty) \times \mathbb{R}^n \mid x_1^2 + |x'|^2 = 1\}$ as the set of vector directions in $\mathbb{R}_1^{n+1}$, where we used the usual notation of Equation (11). Let

$$\langle x \rangle^2 := 1 + \|x\|^2 = \|(1, x)\|^2,$$

as usual. We then have the following simple observation.

Remark 5.1. Let $\Theta_n : \overline{\mathbb{R}^n} = \mathbb{R}^n \sqcup \mathbb{S}_{\mathbb{R}^n} \to \mathbb{S}_1^n$ be given by the formula:

$$\Theta_n(x) := \begin{cases} \frac{1}{\langle x \rangle}(1, x) & \text{if } x \in \mathbb{R}^n, \\ \frac{1}{\|v\|}(0, v) & \text{if } x = \mathbb{R}_+v \in \mathbb{S}_{\mathbb{R}^n}. \end{cases}$$

First, the map Θ_n is well defined because $\mathbb{R}_+v = \mathbb{R}_+w$ implies $v = \lambda w$, for some $\lambda \in \mathbb{R}_+$. Second, Θ_n is $\mathrm{GL}(n, \mathbb{R}) := \mathrm{GL}(\mathbb{R}^n)$-invariant for the action defined in the last paragraph. Finally, it is bijective and its inverse is given by

$$\mathbb{S}_1^n \ni (y_0, y_1, \ldots, y_n) \overset{\Theta_n^{-1}}{\longmapsto} \begin{cases} \frac{1}{y_0}(y_1, \ldots, y_n) \in \mathbb{R}^n & \text{if } y_0 \neq 0 \\ \mathbb{R}_+(y_1, \ldots, y_n) \in \mathbb{S}_{\mathbb{R}^n} & \text{if } y_0 = 0. \end{cases} \tag{30}$$

We endow $\overline{\mathbb{R}^n}$ with the structure of a smooth manifold (with boundary) that makes Θ_n a diffeomorphism. This manifold structure on $\overline{\mathbb{R}^n}$ extends the standard manifold structure of $\mathbb{R}^n$. See also [56, 71].

We now extend the definition of the smooth structure on $\overline{\mathbb{R}^n}$ of Remark 5.1 to any n-dimensional real vector space Z in the usual way. First, choose a vector space isomorphism $Z \to \mathbb{R}^n$, which yields bijections

$$\overline{Z} \overset{\sim}{\longrightarrow} \overline{\mathbb{R}^n} \overset{\sim}{\longrightarrow} \mathbb{S}_1^n. \tag{31}$$

In turn, these bijections can be used to define a smooth structure on the radial compactification $\overline{Z}$ of Z. The $\mathrm{GL}(n, \mathbb{R})$-invariance of Θ_n implies that the resulting smooth structure on $\overline{Z}$ does not depend on the isomorphism $Z \to \mathbb{R}^n$. We note, in passing, that $\overline{Z} \simeq [Z^+ : \{\infty\}]$.

It follows from the definition of the radial compactification and of its topology that, if $Y \subset Z$ is a (linear) subspace, then $\overline{Y} \subset \overline{Z}$ is a closed p-submanifold and $\mathbb{S}_Y = \mathbb{S}_Z \cap \overline{Y}$.

5.2. **Quotients and compactifications.** If Y is a proper linear subspace of X, then the natural projection map $\pi_{X/Y} : X \to X/Y \to \overline{X/Y}$ extends to a well-defined map $\overline{X} \setminus \mathbb{S}_Y \to \overline{X/Y}$, which, at the boundary, is given by $\mathbb{R}_+ x \mapsto \mathbb{R}_+(x + Y)$. This map does not extend to a continuous map on $\overline{X}$, but, as we will show next, it extends to the blow-up of $\overline{X}$ with respect to $\mathbb{S}_Y$.

Proposition 5.2. *The canonical surjection $\pi_{X/Y} : X \to X/Y$ extends to a smooth map $\psi_Y : [\overline{X} : \mathbb{S}_Y] \to \overline{X/Y}$ such that the induced map*

$$\theta_Y := (\beta_{\overline{X}, \mathbb{S}_Y}, \psi_Y) : [\overline{X} : \mathbb{S}_Y] \to \overline{X} \times \overline{X/Y}$$

is a diffeomorphism onto its image, which is a weak submanifold of the product $\overline{X} \times \overline{X/Y}$. Let $G = \mathrm{GL}(X, Y) \subset \mathrm{GL}(X)$ be the group of automorphisms of X that map Y to itself. Then ψ_Y is G-equivariant.

Again, one can check, that the image θ_Y is not a submanifold in the sense of Definition B.1; it is only a weak submanifold (that is, a submanifold in our weaker sense of Definition 2.10.).

Proof. In view of the equivariance of Θ_n and of the bijections in (31), we can assume $X = \mathbb{R}^n$ and $Y = \{0\} \times \mathbb{R}^q$. We will write $\mathbb{S}^{q-1}$ and $\mathbb{R}^q$ instead of $\{0\} \times \mathbb{S}^{q-1}$ and $\{0\} \times \mathbb{R}^q$, for simplicity. Recall that Lemma 4.14 yields a diffeomorphism $\widetilde{\Psi} : [\mathbb{S}^{r,r'}_{k,k'} : \{0\} \times \mathbb{S}^{r'}_{k'}] \xrightarrow{\sim} \mathbb{S}^{r-1}_k \times \mathbb{S}^{r'+1}_{k'+1}$. We shall use this result for $r = n-q+1, r' = q-1, k = 1$, and $k' = 0$. Since $\mathbb{S}^{q-1}_0 = \mathbb{S}^{q-1}$ and $\mathbb{S}^{n-q+1,q-1}_{1,0} = \mathbb{S}^n_1$, we obtain the diffeomorphism

$$\widetilde{\Psi} : [\mathbb{S}^n_1 : \mathbb{S}^{q-1}] \xrightarrow{\sim} \mathbb{S}^{n-q}_1 \times \mathbb{S}^q_1.$$

Let $p_1 : \mathbb{S}^{n-q}_1 \times \mathbb{S}^q_1 \to \mathbb{S}^{n-q}_1$ be the projection onto the first component.

By definition of the smooth structure on $\overline{X}$, the map $\Theta_n : \overline{X} \to \mathbb{S}^n_1 = \mathbb{S}^{n-q+1,q-1}_{1,0}$ of Remark 5.1 is a diffeomorphism, and it maps diffeomorphically $\mathbb{S}_Y$ onto $\mathbb{S}^{q-1}$. Then by Lemma 3.3 we obtain a diffeomorphism $\Theta^\beta_n : [\overline{X} : \mathbb{S}_Y] \to [\mathbb{S}^n_1 : \mathbb{S}^{q-1}]$.

We define ψ_Y as the composition

$$[\overline{X} : \mathbb{S}_Y] \xrightarrow{\Theta^\beta_n} [\mathbb{S}^n_1 : \mathbb{S}^{q-1}] \xrightarrow{\widetilde{\Psi}} \mathbb{S}^{n-q}_1 \times \mathbb{S}^q_1 \xrightarrow{p_1} \mathbb{S}^{n-q}_1 \xleftarrow{\Theta_{n-q}} \overline{X/Y},$$

in other words

$$\psi_Y := (\Theta_{n-q})^{-1} \circ p_1 \circ \widetilde{\Psi} \circ \Theta^\beta_n : [\overline{X} : \mathbb{S}_Y] \to \overline{X/Y},$$

and we claim that ψ_Y is the desired extension.

To prove the claim, recall that we defined $\widetilde{\Psi}$ in Lemma 4.14 as the unique continuous extension of the map

$$\Psi : \mathbb{S}^n_1 \setminus \mathbb{S}^{q-1} \to \mathbb{S}^{n-q}_1 \times \mathbb{S}^q_1, \quad (\eta, \mu) \mapsto \left(\frac{\eta}{|\eta|}, (|\eta|, \mu) \right),$$

where $\eta \in \mathbb{R}^{n-q+1}_1$ and $\mu \in \mathbb{R}^q$. We write $v \in X = Y^\perp \oplus Y$ as $v = (v_\perp, v_Y)$, that is, $v_Y \in Y$ and $v_\perp \perp Y$, which means $v_\perp \in Y^\perp = \mathbb{R}^{n-q} \times \{0\}$. Then, in the case $v_\perp \neq 0$, we have $\Theta_n(v) = \frac{1}{\langle v \rangle}(1, v) \in \mathbb{S}^n_1 \setminus \mathbb{S}^{q-1}$, and in this case we then calculate

$$\widetilde{\Psi} \circ \Theta_n(v) = \Psi\left(\frac{1}{\langle v \rangle}(1, v) \right) = \left(\frac{1}{\langle v_{Y\perp} \rangle}(1, v_{Y\perp}), \frac{1}{\langle v \rangle}(\langle v_{Y\perp} \rangle, v_Y) \right).$$

By continuity of the extension, this formula even holds for all $v \in [\mathbb{S}_1^n : \mathbb{S}^{q-1}]$. By formula (30) we have $\left(\Theta_{n-q}\right)^{-1}(y_0, y_1, \ldots, y_{n-q}) = \frac{1}{y_0}(y_1, \ldots, y_{n-q})$, if $y_0 > 0$. This formula will be used in the following straightforward calculation:

$$\Theta_{n-q}^{-1} \circ p_1 \circ \widetilde{\Psi} \circ \Theta_n(v) = \Theta_{n-q}^{-1}\left(\frac{1}{\langle v_{Y\perp}\rangle}(1, v_{Y\perp})\right) = v_{Y\perp} = \pi_{X/Y}(v).$$

So ψ_Y is indeed the desired extension of $\pi_{X/Y}$.

In the remaining part of the proof, we will use Proposition 2.13 to show that $\theta_Y = (\beta_{\overline{X}, \mathbb{S}_Y}, \psi_Y) : [\overline{X} : \mathbb{S}_Y] \to \overline{X} \times \overline{X/Y}$ is a diffeomorphism on its image, and that the image of this map is a weak submanifold of the product $\overline{X} \times \overline{X/Y}$. One of the conditions required by this proposition is that θ_Y be an injective immersion, which we will check now.

The restriction of the map $\beta_{\overline{X}, \mathbb{S}_Y} : [\overline{X} : \mathbb{S}_Y] \to \overline{X}$ to $\overline{X} \smallsetminus \mathbb{S}_Y$ is a diffeomorphism onto its image, by the definition of the blow-up, and thus $\theta_Y|_{\overline{X} \smallsetminus \mathbb{S}_Y}$ is an injective immersion as well. The complement of $\overline{X} \smallsetminus \mathbb{S}_Y$ in $[\overline{X} : \mathbb{S}_Y]$ is $\beta_{\overline{X}, \mathbb{S}_Y}^{-1}(\mathbb{S}_Y) := \mathbb{S}N_+^{\overline{X}}\mathbb{S}_Y \simeq \mathbb{S}_Y \times \overline{X/Y}$. On this set the map θ_Y becomes the inclusion map

$$\mathbb{S}_Y \times \overline{X/Y} \to \overline{X} \times \overline{X/Y},$$

and obviously this is smooth as well. As θ_Y maps $\overline{X} \smallsetminus \mathbb{S}_Y$ and $\mathbb{S}_Y \times \overline{X/Y}$ to disjoint sets, the injectivity of θ_Y follows. Similarly, the differential of the map θ_Y is also injective at the boundary points. Thus, θ_Y is an injective immersion. Furthermore, θ_Y is defined on a compact set, and thus it is a homeomorphism onto its image. Using Proposition 2.13, we see that its image, $\theta_Y([\overline{X} : \mathbb{S}_Y])$, is a weak submanifold of $\overline{X} \times \overline{X/Y}$, and the diffeomorphism property follows as well. This completes the proof. $\qquad\square$

Remark 5.3. Let $\psi := p_1 \circ \widetilde{\Psi}$, using the notation of the proof of Proposition 5.2. We thus have a commutative diagram

(32)
$$\begin{array}{ccc}
[\overline{X} : \mathbb{S}_Y] & \xrightarrow{\psi_Y} & \overline{X/Y} \\
\Theta_n^\beta \downarrow & & \uparrow \Theta_{n-q}^{-1} \\
[\mathbb{S}_1^{n-1} : \mathbb{S}^{q-1}] & \xrightarrow{\psi} & \mathbb{S}_1^{n-q}
\end{array}$$

The map ψ_Y was also considered in [35, 48].

5.3. Georgescu's constructions using C^*-algebras.

As mentioned before, one of the main motivations of our work is to prove that Georgescu's and Vasy's compactifications of $\mathbb{R}^{3N}$ described in the introduction are canonically homeomorphic. Recall from the Introduction that Georgescu's construction is that of a spectrum of a commutative C^*-algebra [32, 34, 35], whereas Vasy used blow-ups [71, 72]. Georgescu's construction provides a topological space, whereas Vasy's construction defines a smooth manifold with corners. Thus a homeomorphism of these spaces that extends the identity of $\mathbb{R}^{3N}$ is the best that we can hope for. In turn, however, this homeomorphism will then equip Georgescu's compactification with the structure of a smooth manifold with corners. To compare the approaches of these two authors, we need to recall a few facts about commutative C^*-algebras. We refer to [29, 67] for more basic facts about C^*-algebras.

Definition 5.4. A C^*-*algebra* A is an algebra over $\mathbb{C}$ with a norm $\|.\|$ and with a map $* : A \to A$ such that A is a Banach algebra and for every $\lambda, \mu \in \mathbb{C}$ and $a, b \in A$, we have

(i) $(a^*)^* = a,$

(ii) $(ab)^* = b^* a^*$,
(iii) $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$,
(iv) $\|a a^*\| = \|a\|^2$.

The C^*-algebra is *commutative* if $ab = ba$ for all $a, b \in A$.

Every closed, self-adjoint subalgebra of bounded operators on a Hilbert space is a C^*-algebra. In fact, this is a general example, as a basic result is that every C^*-algebra is isometrically isomorphic to a closed, self-adjoint subalgebra of bounded operators on a Hilbert space. We shall mostly be interested in the following commutative C^*-algebra.

Let us fix from now on a finite semilattice $\mathcal{F}$ of linear subspaces of some finite dimensional, real vector space X. It will be convenient to assume that $X \notin \mathcal{F}$, but that $\{0\} \in \mathcal{F}$. Our approach works also without these assumptions, but they do simplify the presentation.

Example 5.5. Give two vector spaces X and $Y \subset X$, the composition

$$X \xrightarrow{\pi_{X/Y}} X/Y \xrightarrow{\text{incl}} \overline{X/Y}$$

induces by pullback an injective map $\mathcal{C}(\overline{X/Y}) \xrightarrow{\pi^*_{X/Y}} \mathcal{C}_b(X)$, where $\mathcal{C}_b(X)$ is the C^*-algebra of continuous and bounded complex-valued functions on X, again equipped with the supremum norm. Recall that $\mathcal{F}$ be a finite semilattice of linear subspaces of X, $X \notin \mathcal{F}$, $\{0\} \in \mathcal{F}$. As in the Introduction, Equation (7), let $\mathcal{E}_{\mathcal{F}}(X)$ be the norm closed subalgebra of $\mathcal{C}_b(X)$ generated by the pullbacks of the spaces $\mathcal{C}(\overline{X/Y})$, where Y runs over $\mathcal{F}$. Then $\mathcal{E}_{\mathcal{F}}(X)$ is stable for complex conjugation, and hence it is a C^*-algebra that contains $\mathcal{C}(\overline{X})$ because $\{0\} \in \mathcal{F}$.

Another general result is that all commutative C^*-algebra are isometrically isomorphic to one of the form $\mathcal{C}_0(Z)$, discussed in the following example.

Example 5.6. For a locally compact and Hausdorff topological space Z, let $\mathcal{C}_0(Z)$ be the algebra of complex-valued continuous function f on Z that vanish at infinity (in the sense that the set $|f(z)| \geq \epsilon > 0$ is compact for all $\epsilon > 0$). We endow $\mathcal{C}_0(Z)$ with the involution $f^* = \overline{f}$ (the complex conjugation) and with the norm $\|f\|_\infty = \sup_{z \in Z} |f(z)|$. With this structure, $\mathcal{C}_0(Z)$ is a commutative C^*-algebra. It is unital if, and only if, Z is compact.

The space Z in the last example can be recovered (up to a homeomorphism) from the algebra $\mathcal{C}_0(Z)$ via *characters*, so we now recall this concept. A *character* of a C^*-algebra $\mathcal{A}$ is a non-zero $*$-morphism $\chi : \mathcal{A} \to \mathbb{C}$. A basic result is that such a character is continuous of norm 1. If $\mathcal{A}$ is commutative, we denote by $\mathrm{Spec}(\mathcal{A}) \subset \mathcal{A}^*$ the set of characters of $\mathcal{A}$ and endow it with the topology induced from the weak topology on $\mathcal{A}^*$. It is a locally compact space and $\mathcal{A} \simeq \mathcal{C}_0(\mathrm{Spec}(\mathcal{A}))$. If $\mathcal{A} = \mathcal{C}_0(Z)$, then Z and $\mathrm{Spec}(\mathcal{A})$ are homeomorphicm via evaluations: $Z \ni z \to \mathrm{ev}_z \in \mathrm{Spec}(\mathcal{A}) \subset A^*$, $\mathrm{ev}_z(f) := f(z)$. Although we shall not use this in this paper, let us mention that, in case $\mathcal{B}$ is non-commutative, it may have very few characters (maybe none!), so the concept of $\mathrm{Spec}(\mathcal{B})$ is not very useful in this case. It is rather the concept of a primitive ideal spectrum that is useful. Recall that an ideal of $\mathcal{B}$ is *primitive* if it is the kernel of a non-zero irreducible representation of $\mathcal{B}$. If $\mathcal{A}$ is commutative, then the primitive ideals of $\mathcal{A}$ are exactly the maximal ideal of $\mathcal{A}$ and there is a one-to-one correspondence between the set of characters $\chi : \mathcal{A} \to \mathbb{C}$ of A and the set of maximal ideals of A. This correspondence is given by $\chi \mapsto \ker(\chi)$.

In view of the discussion and the examples just introduced, we can now recall Georgescu's definition of the compactification of X.

Definition 5.7. Let $\mathcal{F}$ be a finite semilattice of linear subspaces of the finite dimensional vector space X with $\{0\} \in \mathcal{F}$ and $X \notin \mathcal{F}$, as (as agreed in the lines before Example 5.5). Then the spectrum $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ of the algebra introduced in Equation (7) is called *Georgescu's compactification of X with respect to $\mathcal{F}$.*

This definition makes sense since $\mathcal{C}_0(X) \subset \mathcal{E}_{\mathcal{F}}(X)$, by definition (since we have assumed that $\{0\} \in \mathcal{F}$, and this assumption was introduced exactly for this reason.) In [61, Theorem 4.4], two of the authors of this paper (together with Prudhon) have proved the following result. Recall the space $X_{\mathrm{GV}} := \overline{\delta_{\mathcal{F}}(X)}$ defined in Equation (6).

Proposition 5.8. *The spectrum $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ of $\mathcal{E}_{\mathcal{F}}(X)$ is homeomorphic to the closure $X_{\mathrm{GV}} := \overline{\delta_{\mathcal{F}}(X)}$ of the image of X in the product $\prod_{Y \in \mathcal{F}} \overline{X/Y}$ under the "diagonal" map $\delta_{\mathcal{F}} : X \to \prod_{Y \in \mathcal{F}} \overline{X/Y}$, $\delta_{\mathcal{F}}(x) := (\pi_Y(x))_{Y \in \mathcal{F}}$. More precisely, the homeomorphism $\Phi_{\mathcal{F}} : \overline{\delta_{\mathcal{F}}(X)} \to \mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ is given as follows. Let $z = (z_Y)_{Y \in \mathcal{F}}$ be in the closure of $\delta_{\mathcal{F}}(X)$. Then the homeomorphism $\Phi_{\mathcal{F}}$ sends z to the character χ_z defined by $\chi_z(f_Y) = f_Y(z_Y)$ whenever $f_Y \in \mathcal{C}(\overline{X/Y})$.*

Proposition 5.8 thus identifies the spectrum $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ of the C^*-algebra $\mathcal{E}_{\mathcal{F}}(X)$ (Georgescu's space) with the space $X_{\mathrm{GV}} := \overline{\delta_{\mathcal{F}}(X)}$ introduced in [61] and recalled in Equation (6).

5.4. **Identification of the Georgescu and Vasy spaces.** Recall that beginning with Example 5.5, we have assumed that $\mathcal{F}$ denotes a finite semilattice of linear subspaces of X with $\{0\} \in \mathcal{F}$, $X \notin \mathcal{F}$. Let $\mathbb{S}_{\mathcal{F}} := \{\mathbb{S}_Y = \mathbb{S}_X \cap \overline{Y} \mid Y \in \mathcal{F}\}$ be the semilattice introduced in Equation (8). Then $\emptyset \in \mathbb{S}_{\mathcal{F}}$, as it corresponds to the subspace $\{0\} \subset X$ that was assumed to be in $\mathcal{F}$. Moreover, $\mathbb{S}_{\mathcal{F}}$ is a clean semilattice and we endow it with an admissible order. Recall also the space $X_{\mathrm{GV}} := \overline{\delta_{\mathcal{F}}(X)}$ introduced in Equation (6) in the Introduction and the fact that the graph blow-up $\{\overline{X} : \mathbb{S}_{\mathcal{F}}\}$ has a natural structure of manifold with corners, by Theorem 4.19.

Proposition 5.9. *The product map*

$$\Psi_{\mathcal{F}} := \prod_{Y \in \mathcal{F}} \psi_Y : \prod_{Y \in \mathcal{F}} [\overline{X} : \mathbb{S}_Y] \to \prod_{Y \in \mathcal{F}} \overline{X/Y}$$

of the maps ψ_Y of Proposition 5.2 induces a diffeomorphism of the graph blow-up $\{\overline{X} : \mathbb{S}_{\mathcal{F}}\} \subset \prod_{Y \in \mathcal{F}} [\overline{X} : \mathbb{S}_Y]$ onto its image. Moreover, its image is $\overline{\delta_{\mathcal{F}}(X)} =: X_{\mathrm{GV}}$, so the latter is a weak submanifold of the product, and hence a manifold with corners on its own.

Proof. Let

$$\Theta_{\mathcal{F}} := \prod_{Y \in \mathcal{F}} \theta_Y : \prod_{Y \in \mathcal{F}} [\overline{X} : \mathbb{S}_Y] \to \prod_{Y \in \mathcal{F}} (\overline{X} \times \overline{X/Y}),$$

be the product of the maps θ_Y of Proposition 5.2. By that proposition, the map $\Theta_{\mathcal{F}}$ is a product of injective immersions, and hence it is an injective immersion itself. Hence $\Theta_{\mathcal{F}}$ maps $\{\overline{X} : \mathbb{S}_{\mathcal{F}}\}$ diffeomorphically onto its image. The restriction of the map $\Theta_{\mathcal{F}}$ to $\{\overline{X} : \mathbb{S}_{\mathcal{F}}\}$ repeats the component corresponding to $\overline{X}$, and hence, it is obtained from $\Psi_{\mathcal{F}} := \prod_{Y \in \mathcal{F}} \psi_Y$ by repeating these components. (Note that if $Y = \{0\}$, then $\psi_Y = \psi_{\{0\}} = \mathrm{id} : \overline{X} \to \overline{X} = \overline{X/Y}$, so all the $\overline{X}$ component can be obtained from $\psi_{\{0\}}$, and hence from $\Psi_{\mathcal{F}} : \prod_{Y \in \mathcal{F}} \psi_Y$ as well. (This is yet another reason why we assume $\{0\} \in \mathcal{F}$.) It follows that $\Psi_{\mathcal{F}}$ also maps $\{\overline{X} : \mathbb{S}_{\mathcal{F}}\}$ diffeomorphically onto its image in $\prod_{Y \in \mathcal{F}} \overline{X/Y}$. The result follows since X is dense in both $\{\overline{X} : \mathbb{S}_{\mathcal{F}}\}$ and in $X_{\mathrm{GV}} := \overline{\delta_{\mathcal{F}}(X)}$. $\square$

We now obtain the desired diffeomorphism between Vasy's space $\left[\overline{X} : \mathbb{S}_{\mathcal{F}}\right]$ with the Georgescu-Vasy space $X_{\mathrm{GV}} := \overline{\delta_{\mathcal{F}}(X)}$ introduced in [61] (see Equation (6)).

Proposition 5.10. *The product map*

(33) $$\Xi_{\mathcal{F}} : \left[\overline{X} : \mathbb{S}_{\mathcal{F}}\right] := \prod_{Y \in \mathcal{F}} \psi_Y \circ \phi_{\mathbb{S}_{\mathcal{F}}, \mathbb{S}_Y} \ \rightarrow \ \prod_{Y \in \mathcal{F}} \overline{X/Y}$$

of the composite maps

$$\left[\overline{X} : \mathbb{S}_{\mathcal{F}}\right] \xrightarrow{\phi_{\mathbb{S}_{\mathcal{F}}, \mathbb{S}_Y}} \left[\overline{X} : \mathbb{S}_Y\right] \xrightarrow{\psi_Y} \overline{X/Y}$$

is a diffeomorphism onto $X_{\mathrm{GV}} := \overline{\delta_{\mathcal{F}}(X)}$. *Let G be a Lie group of linear automorphisms of X that map elements of $\mathcal{F}$ to elements of $\mathcal{F}$ (thus $g(\mathbb{S}_{\mathcal{F}}) = \mathbb{S}_{\mathcal{F}}$ for all $g \in G$). Then G acts smoothly on $\left[\overline{X} : \mathbb{S}_{\mathcal{F}}\right]$, the map $\Xi_{\mathcal{F}}$ is G-equivariant, and G acts smoothly on X_{GV}.*

For $Y = \{0\}$, we have $\mathbb{S}_Y = \emptyset$ (yet another reason for requiring $\{0\} \in \mathcal{F}$ and $\emptyset$ to belong to our semilattices), and hence the map $\psi_Y \circ \phi_{\mathbb{S}_{\mathcal{F}}, \mathbb{S}_Y} = \psi_Y \circ \phi_{\mathbb{S}_{\mathcal{F}}, \emptyset}$ is simply the blow-down map $\left[\overline{X} : \mathbb{S}_{\mathcal{F}}\right] \to \overline{X}$.

Proof. This follows by combining Proposition 5.9 with Theorem 4.19 applied to the semilattice $\mathbb{S}_{\mathcal{F}}$ of closed p-submanifolds of $\overline{X}$ (so $\mathcal{S}$ of that theorem is replaced by $\mathbb{S}_{\mathcal{F}}$). More precisely, in that theorem, the pairs $(\mathcal{S}, P_j)$ are replaced with the pairs $(\mathbb{S}_{\mathcal{F}}, \mathbb{S}_Y)$, $Y \in \mathcal{F}$, and the map $\mathcal{B}_{\mathcal{S}}$ is replaced with the map $\mathcal{B}_{\mathbb{S}_{\mathcal{F}}} := \prod_{Y \in \mathcal{F}} \phi_{\mathbb{S}_{\mathcal{F}}, \mathbb{S}_Y}$. Thus Theorem 4.19 gives a diffeomorphism $\mathcal{B}_{\mathbb{S}_{\mathcal{F}}} : \left[\overline{X} : \mathbb{S}_{\mathcal{F}}\right] \to \{\overline{X} : \mathbb{S}_{\mathcal{F}}\}$ (including a manifold with corners structure on the latter). The fact that $\Xi_{\mathcal{F}}$ is a diffeomorphism with the stated properties follows from the diffeomorphism $\Psi_{\mathcal{F}} : \{\overline{X} : \mathbb{S}_{\mathcal{F}}\} \to X_{\mathrm{GV}}$ of Proposition 5.9 and the fact that $\Xi_{\mathcal{F}} = \Psi_{\mathcal{F}} \circ \mathcal{B}_{\mathbb{S}_{\mathcal{F}}}$.

Finally, the action of G and the fact that $\Xi_{\mathcal{S}}$ is G-equivariant follow from the fact that all the maps used to definite $\Xi_{\mathcal{S}}$ are G-equivariant and from Proposition 3.10. $\square$

Combining Propositions 5.8 and 5.10, we obtain the following result.

Theorem 5.11. *Let $\mathcal{F}$ be a finite semilattice of linear subspaces of X containing $\{0\}$ and $\mathbb{S}_{\mathcal{F}} := \{\mathbb{S}_Y \mid Y \in \mathcal{F}\}$ be as in Equation (8). There exists a unique homeomorphism*

$$\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X)) \ \simeq \ \left[\overline{X} : \mathbb{S}_{\mathcal{F}}\right]$$

that is the identity on X.

Proof. Let $\delta_{\mathcal{F}} : X \to \prod_{Y \in \mathcal{F}} \overline{X/Y}$ be the diagonal map. Proposition 5.8 states that we have a homeomorphism $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X)) \to X_{\mathrm{GV}} := \overline{\delta_{\mathcal{F}}(X)}$. The result follows from Proposition 5.10, which states that the map $\Xi_{\mathcal{F}}$ defined on $\left[\overline{X} : \mathbb{S}_{\mathcal{F}}\right]$ is a diffeomorphism onto X_{GV}. $\square$

To conclude, the above results show that the following spaces:

- the iterated blow-up $\left[\overline{X} : \mathbb{S}_{\mathcal{F}}\right]$ (Vasy's space),
- the graph blow-up $\{\overline{X} : \mathbb{S}_{\mathcal{F}}\}$,
- $X_{\mathrm{GV}} := \overline{\delta_{\mathcal{F}}(X)}$ of Equation (6), and
- $\mathrm{Spec}(\mathcal{E}_{\mathcal{F}}(X))$ (Georgescu's space)

are all homeomorphic. This yields the sequence of homeomorphisms (9) of the Introduction. More precisely, we can complete that equation with the explicit diffeomorphisms

proved (in order) in Theorem 4.19 (for $\mathcal{B} := \mathcal{B}_{\mathbb{S}_\mathcal{F}}$), Proposition 5.9 (for $\Psi := \Psi_\mathcal{F}$), and, finally, Proposition 5.8 for the last morphism $\Phi := \Phi_\mathcal{F}$

$$(34) \qquad [\overline{X} : \mathbb{S}_\mathcal{F}] \xrightarrow{\mathcal{B}} \{\overline{X} : \mathbb{S}_\mathcal{F}\} \xrightarrow{\Psi} X_{\mathrm{GV}} := \overline{\delta_\mathcal{F}(X)} \xrightarrow{\Phi} \mathrm{Spec}(\mathcal{E}_\mathcal{F}(X)) \,.$$

Any of these spaces will be denoted X_{GV} from now on and called the *Georgescu-Vasy space*. We obtain as a corollary the following description for the space introduced in [34, 35] (the "small Georgescu space").

Remark 5.12. In [34, 35], Georgescu and his collaborators have considered the norm closed subalgebra of functions $\mathfrak{A}_\mathcal{F}$ of $L^\infty(X)$ generated by all the algebras $\mathcal{C}_0(X/Y)$ with $Y \in \mathcal{F}$. This corresponds to potentials that have zero limit at infinity on X/Y. The spectrum of this algebra (after adjoining a unit) identifies with the closure of the image of the diagonal map of X to $\prod_{Y \in \mathcal{S}}(X/Y)^+$, where Z^+ denotes the one point compactification of a locally compact space Z. (This is a result analogous to Proposition 5.8, likewise proved in [61].) Since $\mathfrak{A}_\mathcal{F} \subset \mathcal{E}_\mathcal{F}(X)$, we obtain that $\mathrm{Spec}(\mathfrak{A}_\mathcal{F})$ is a quotient of $\mathrm{Spec}(\mathcal{E}_\mathcal{F}(X))$, and hence also a quotient of

$$X_{\mathrm{GV}} := [\overline{X} : \mathbb{S}_\mathcal{F}],$$

by Theorem 5.11. Generally, the topology on $\mathrm{Spec}(\mathfrak{A}_\mathcal{F})$ is rather complicated and singular, see also [55, Section 5] for concrete examples when $\dim(X) = 2$.

6. APPLICATIONS TO THE *N*-BODY PROBLEM

Our main motivation is that, by identifying the spaces appearing in Georgescu's and Vasy's constructions (Theorem 5.11), one will be able to combine the results and the techniques in their papers and in other related papers to obtain new results. (Among the papers that we have in mind are Georgescu's works [6, 20, 32, 34, 35] and Vasy's papers [71, 72] as well as in [25, 48, 49, 64, 65], and in the references therein). In this spirit, in this section, we discuss some applications of our results. We provide a brief, but complete account of these applications based on a complete set of references.

6.1. **The *N*-body semilattice and Pauli exclusion principle.** The setting considered in the previous sections of a semilattice $\mathcal{F}$ of linear subspaces of a vector space X is inspired from the *N*-body problem. In this subsection, we explain the concrete choice of $\mathcal{F} = \mathcal{F}_N$ and of X in the case of the *N*-body problem and notice that it is compatible with symmetry and antisymmetry assumptions as, for instance, the Pauli exclusion principle. More precisely, the Georgescu-Vasy space associated to the semilattice of the effective *N*-body problem carries a natural, concrete action of the symmetric group S_N (the permutation group on N letters). This subsection, while relevant on its own, also sets the stage for the applications in the following sections.

6.1.1. *The semilattice of the N-body problem.* Here is what the choices of X and $\mathcal{F}$ are for the effective Hamiltonian H_{N-1}^{eff} of the *N*-body problem.

Example 6.1. In the concrete case of the Hamiltonian H'_N of Equation (3), we take $X := \mathbb{R}^{3N}$ and consider the subspaces

$$Y_j := \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^{3N} \mid x_j = 0\} \quad \text{and}$$
$$Y_{ij} := \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^{3N} \mid x_j = x_j\}, \ i \neq j \,.$$

Thus each $x_i \in \mathbb{R}^3$. We let $\mathcal{F} := \mathcal{F}_N$, be the semilattice generated by the subspaces Y_i and Y_{ij}, $i, j \in \{1, 2, \dots, N\}$ [6, 25]. (This example is related to the Born-Oppenheimer approximation for a system with a single nucleus [42].) The case of H_N of Equation

(1) is very similar: we only consider the subspaces Y_{ij}. We note, however, that, in this case $\{0\}$ will not be in the semilattice generated, but the minimal element is the subspace $\{(x, x, \ldots, x) \mid x \in \mathbb{R}^3\}$. That is not a real problem, however, since the condition $\{0\} \in \mathcal{F}$ is imposed only for convenience. Besides, one can always increase $\mathcal{F}$ by including also the zero subspace. This problem does not arise in the case of H_N' or H_{N-1}^{eff}. For H_{N-1}^{eff}, the semilattice would be more difficult to describe. See the last chapter of [6] for a complete treatement of this class of examples.

In particular, our results give the following.

Remark 6.2. Let us consider the case of the Hamiltonian H_N' of Equation (3), the case of the usual N-body Hamiltonian H_N being completely similar. Let the vector space be $X := \mathbb{R}^{3N}$ and let the semilattice $\mathcal{F} := \mathcal{F}_N$ be as in the last example, Example 6.1. Let $\mathbb{S}_{\mathcal{F}_N} := \{\mathbb{S}_Y \mid Y \in \mathcal{F}_N\}$ be the finite semilattice of closed p-submanifolds of $\overline{X}$ as in Equation (8). Then our results, especially Theorem 4.21 imply that $M_N := [\overline{X} : \mathbb{S}_{\mathcal{F}_N}] = X_{\mathrm{GV}}$, the Georgescu-Vasy space associated to the semilattice $\mathcal{F}_N$, will be endowed with natural, smooth actions of the following groups:

- S_N, the symmetric group on $\{1, 2, \ldots, N\}$, acting on the variables by permutation;
- $\mathrm{GL}(3, \mathbb{R})$ acting diagonally on the components of $X := \mathbb{R}^{3N}$; and
- X, extending the action by translation on itself. (This is valid for all semilattices $\mathcal{F}$ of linear subspaces of X, not just for $\mathcal{F}_N$, yielding a smooth action of X on X_{GV}).

(These actions can also be obtained from Theorem 5.11 and Proposition 5.10.) These actions are easy to obtain at the level of spectra of C^*-algebras or for the graph-family blow-up, but more difficult to obtain geometrically using iterated blow-ups. In particular, in [59], it was formulated the problem of constructing a compactification of $\mathbb{R}^{3N}$ endowed with the action of the symmetric group as above. Answers to this problem were provided in [48, 62]. The smoothness of these actions is based on Theorem 5.11, since the groups act smoothly on each $\overline{X/Y}$, $Y \in \mathcal{F}$.

See also [11, 12, 13, 18, 21, 33, 36, 37, 44, 45] for physically relevant results that can point out to further extensions of our work, including to Quantum Field Theory on a curved space-time.

6.1.2. *Symmetry, antisymmetry and the Pauli exclusion principle.* As already remarked above, the action of the symmetric group S_N on $M_N := [\overline{X} : \mathbb{S}_{\mathcal{F}_N}]$ is important for applications. Recall that, in the motivational part of the Introduction, we allowed mixed systems of particles. Some of them will be bosons, in which case the wave function will be symmetric under permuation of two variables corresponding to bosons of the same kind. Other particles will be fermions (for instance, electrons), in which case the wave function is antisymmetric under permutations of two variables corresponding to fermions of the same kind. This is commonly known as the *Pauli exclusion principle*. In total, we consider the subgroup $\Gamma \subset S_N$ of permutations of particles of the same kind, and we obtain a map $\chi : \Gamma \to \{-1, +1\}$ such that only functions with

$$f : \mathbb{R}^{3N} \to \mathbb{C}, \quad f(gx) = \chi(g)f(x)$$

are allowed as wave functions for physical reason, where Γ acts on $\mathbb{R}^{3N}$ by permutation of components. It is thus helpful that we have proven, see Theorem 4.21, that this Γ-actions extends to the compactification M_N.

In fact the situation becomes slightly more complicated if some particles will have spin, which implies – in mathematical terms – that they are vector valued. As an example, which hopefully is representative of the general case, let us explain the case of N electrons. As electrons have spin $1/2$, we should enlarge the target of the wave function and discuss functions

$$\Psi : \mathbb{R}^{3N} \to (\mathbb{C}^2)^{\otimes N}.$$

Here the tensor product is the tensor product over $\mathbb{C}$ and the k-th factor models the spin of the k-th electron. Let S_N act on the target $(\mathbb{C}^2)^{\otimes N}$ by permutation of the factors, that is, $g(v_1 \otimes \cdots \otimes v_N) = (v_{g(1)} \otimes \cdots \otimes v_{g(N)})$ and on $\mathbb{R}^{3N}$ be exchanging the component vectors, more precisely, $g(x_1, \ldots, x_N)$ to $(x_{g(1)}, \ldots, x_{g(N)})$. The Pauli exclusion principle states that the physically allowed wave functions are described by functions satisfying

$$\Psi(g(x)) = \operatorname{sgn}(g) g(\Psi(x)).$$

6.2. **Vasy's pseudodifferential calculus and Georgescu's algebra.** We will now return to the more general setting of a general semilattice $\mathcal{F}$ of linear subspaces of a finite-dimensional vector space X, using the notation of (8). Recall that we are assuming, for convenience, that $\{0\} \in \mathcal{F}$ and $X \notin \mathcal{F}$ (this is no loss of generality, since our argument works in general but is just a little bit more involved; moreover, the general case can be reduced to this one).

The action of X by translation on

$$(35) \qquad\qquad\qquad\qquad X_{\mathrm{GV}} = [\overline{X} : \mathbb{S}_{\mathcal{F}}]$$

(see Remark 6.2 and Theorem 4.21) can be used to define Georgescu's algebra and (possibly) Vasy's pseudodifferential calculus, along the lines of Georgescu's method [32, 34] (see also [4]). Let us outline this construction and derive some consequences.

Let $\mathscr{S}(X)$ denote the Schwartz space of smooth, rapidly decreasing functions on X. Any $f \in \mathscr{S}(X)$ gives rise to a convolution operator $f(T) : L^2(X) \to L^2(X)$, $h \mapsto f * h$. In the notation $T : X \to \mathcal{L}(L^2(X))$, $q \mapsto T_q$ stands for the translation operator $T_q f(x) = f(x+q)$ as, for instance, in [34, 35]. In fact, much more general functions f can be allowed here, such as function whose Fourier transform is a classical symbol. Similarly, a function $g \in \mathcal{C}(X_{\mathrm{GV}})$ gives rise to a multiplication operator M_g on $L^2(X)$. By results of Georgescu [34] (using also Theorem 5.11), Georgescu's algebra $\mathcal{C}(X_{\mathrm{GV}}) \rtimes X$ is the norm closure of the algebra generated by operators of the form $M_g f(T)$ acting on $L^2(X)$. So, if $\mathcal{L}(\mathcal{H})$ denotes the algebra of bounded operators on a Hilbert space $\mathcal{H}$, then we obtain $\mathcal{C}(X_{\mathrm{GV}}) \rtimes X \subset \mathcal{L}(L^2(X))$. The resulting subalgebra $\mathcal{C}(X_{\mathrm{GV}}) \rtimes X$, called crossed product, is norm closed and closed under adjoints, hence is a C^*-algebra. See [34, 32] for the details on the link between the crossed product by $\mathbb{R}^n$ of such a commutative C^*-algebra and operators of the form $M_g f(T)$.

If f is such that its Fourier transform is a classical symbol of order m on X, then $P := M_g f(T)$ is a pseudodifferential operator. Classically then, its distribution kernel $k_P \in \mathcal{D}'$ is a classical conormal distribution in $I^m(X \times X; X)$, with X diagonally embedded in $X \times X$. (See [39] for the definition of (classical) conormal distributions.) The map $(x_1, x_2) \to x_1 - x_2$ extends then to a smooth map of pairs $(X \times X, X) \to (X_{\mathrm{GV}} \times X; X_{\mathrm{GV}})$, with the embedding $X_{\mathrm{GV}} \simeq X_{\mathrm{GV}} \times \{0\} \subset X_{\mathrm{GV}} \times X$. This embedding sends k_P to $g \otimes f$, and hence k_P can be identified with a classical conormal distribution in $I^m(X_{\mathrm{GV}} \times X; X_{\mathrm{GV}})$ (this is a particular case of the construction in [4]). Let $I^m_c(X_{\mathrm{GV}} \times$

$X; X_{\mathrm{GV}})$ be the set of such distributions with compact support. Then it follows that

$$\Psi_c^\infty(X_{\mathrm{GV}}) := I_c^\infty(X_{\mathrm{GV}} \times X; X_{\mathrm{GV}}) := \bigcup_{m \in \mathbb{Z}} I_c^m(X_{\mathrm{GV}} \times X; X_{\mathrm{GV}})$$

is a filtered algebra acting by convolution on (suitable) functions $X \to \mathbb{C}$ as an algebra of pseudodifferential operators [4]. (Recall that we are considering only classical conormal distributions and the index "c" comes from "compact support." Vasy's N-body calculus $\Psi_N^\infty(X)$ is certainly bigger and better than $I_c^\infty(X_{\mathrm{GV}} \times X; X_{\mathrm{GV}})$ in the sense that it contains the resolvents of its L^2-invertible operators. Let

$$I_c^{-\infty}(X_{\mathrm{GV}} \times X; X_{\mathrm{GV}}) \subset \mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{\mathrm{GV}}) \subset I^{-\infty}(X_{\mathrm{GV}} \times X; X_{\mathrm{GV}})$$

be the projective tensor product. We have good reasons to believe and hence we conjecture that Vasy's N-body calculus can be identified with

$$(36) \qquad \Psi_{NB}^\infty(X_{\mathrm{GV}}) := I_c^\infty(X_{\mathrm{GV}} \times X; X_{\mathrm{GV}}) + \mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{\mathrm{GV}}) \,.$$

We need to include $\mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{\mathrm{GV}}) := \mathscr{S}(X; \mathcal{C}^\infty(X_{\mathrm{GV}}))$ on the right hand side to accomodate operators of the form $M_g f(T)$ with $f \in \mathscr{S}(X)$ *with non-compact support*, since $M_g f(T) \in I_c^m(X_{\mathrm{GV}} \times X; X_{\mathrm{GV}})$ if, and only if, f is compactly supported (recall that $\hat{f}$ is a classical symbol of order m).

Proposition 6.3. *We define*

$$\Psi_{NB}^n(X_{\mathrm{GV}}) := I_c^n(X_{\mathrm{GV}} \times X; X_{\mathrm{GV}}) + \mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{\mathrm{GV}}) \,.$$

The space $\Psi_{NB}^\infty(X_{\mathrm{GV}}) := \bigcup_k \Psi_{NB}^k(X_{\mathrm{GV}})$ *is a filtered algebra that is closed under holomorphic functional calculus. Let D be a strongly elliptic differential operator of order $m > 0$, with constant coefficients and $v_Y \in \mathcal{C}^\infty(\overline{X/Y})$. Then $H_N' := D + \sum_{Y \in \mathcal{F}} v_Y \in \Psi_{NB}^m(X_{\mathrm{GV}})$. Consequently, for all $\lambda \notin \mathrm{Spec}(H_N')$, we have*

$$(H_N' - \lambda)^{-1} \in \Psi_{NB}^{-m}(X_{\mathrm{GV}}) := I_c^{-m}(X_{\mathrm{GV}} \times X; X_{\mathrm{GV}}) + \mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{\mathrm{GV}}) \,.$$

In the case $X := \mathbb{R}^{3N}$ and $\mathcal{F}$ ans in the N-body problem, the action of the symmetric group S_N on X induces an order-preserving automorphism of the algebra $\Psi_{NB}^\infty(X_{\mathrm{GV}})$.

Proof. (Sketch) There are two main things to prove here: first, that the convolution product makes $\mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{\mathrm{GV}}) := \mathscr{S}(X; \mathcal{C}^\infty(X_{\mathrm{GV}}))$ an algebra and, second, that it is stable for holomorphic functional calculus (equivalently in this case, that the algebra with adjoint unit contatins the resolvents of its L^2-invertible elements). The first question is answered by noticing that the action of X on $\mathcal{C}^\infty(\overline{X/Y})$ is with polynomial growth (this is quite unusual for the action of X on a manifold!) and hence it is again with polynomial growth on X_{GV} in view of our Theorem 5.11. The second question is answered by using the results of [51] as follows. We consider three families of operators on $L^2(X_{\mathrm{GV}} \times X)$, possible unbounded (so not defined everywhere). Let A_1 be the set of differential operators on X_{GV}, let A_2 the set of multiplication operators with polynomials on X and, finally, let A_3 be the set of constant coefficients differential operators on X. Then

$$\mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{\mathrm{GV}}) := \left\{ f \in \mathcal{C}(X_{\mathrm{GV}}) \rtimes X \,\Big|\, \big[[f, P_1], P_2\big] P_3 \text{ is bounded } P_j \in A_j \right\} \,.$$

The results of [51], especially Theorems 2 and 3, then give that $\mathscr{S}(X) \otimes_\pi \mathcal{C}^\infty(X_{\mathrm{GV}})$ is spectrally invariant (*i. e.,* stable under holomorphic functional calculus). $\qquad\square$

We ignore if one can replace in the resolvent estimate of the last proposition H_N' with H_N or with H_{N-1}^{eff}, which are not in $\Psi_{NB}^\infty(X_{\mathrm{GV}})$, since these operators allow for Coulomb singularities in the potential. This brings us to one of our main reasons for considering

Georgescu's algebras $\mathcal{E}_{\mathcal{F}}(X) \rtimes X$ instead of a pseudodifferential calculus (and one of the reasons why we may need to take norm closures), namely, that Georgescu's algebra does not suffer from this deficiency, and, in fact, one has

$$(37) \qquad (H_N - \lambda)^{-1} \in \mathcal{E}_{\mathcal{F}}(X) \rtimes X, \quad \lambda \notin \mathrm{Spec}(H_N).$$

[6, 34, 32] (this is a consequence of Hardy's inequality and is explained also in [35]). Of course, the above proposition provides a much more precise result, when applicable, but is also much more difficult to prove than the relation of Equation (37). Let us notice, moreover, that $\mathcal{E}_{\mathcal{F}}(X) \rtimes X$ is the norm closure of $\Psi_{NB}^{-1}(X_{\mathrm{GV}})$ in $\mathcal{L}(L^2(X))$, the algebra of bounded operator on $L^2(X)$.

6.3. **Connections to the HVZ theorem.** The algebras considered in the previous subsection were introduced, in part, in order to obtain conceptual proofs and extensions of the classical HVZ theorem, named after Hunziker, van Winter, and Zhislin, describing the essential spectrum of N-body Hamiltonians H [25, 26, 32, 65, 69]. It is well-known that the operators H considered here are self-adjoint. We have that λ *is not* in the essential spectrum of H if, and only if, $H - \lambda$ is Fredholm. Our next application is of a conceptual nature on how to relate the HVZ theorem with other classical Fredholm results in PDE theory. There exist many refinements of the HVZ theorem, in the simple setting of an atom with N-electrons, we refer to [69, Section 11, Theorem 11.2]; for a more general version see [68, Theorem XIII.12]. The HVZ theorem determines the essential spectrum of the Hamiltonian H_N in terms of other, simpler Hamiltonians H_{N_α}, where α ranges over a certain index set. The operators H_{N_α} are usually called "limit operators," and can indeed be obtained as strong limits of translations of H_N. Very powerful generalizations of the HVZ theorem were obtained by Georgescu (using C^*-algebras [6, 32, 35]) and by many other authors – more on this below.

Nowadays, there are many results telling us when (pseudo)differential operators on non-compact or singular spaces are Fredholm, and typically they are also in terms of certain "generalized limit operators", the terminologies "indicial operator" or "normal symbol" are also used by Melrose and Schulze independently. We refer to [52] for a overview and comparison between these two approach. Results of this type go back at least to Kondratiev's 1967 celebrated paper [46]. Some of the strongest current results are based on groupoids, see [16, 17, 63] and the references therein. We also refer to [19, 65] for the case when the groupoid is obtained from the action of a group on a space, as it is our case in this paper. The results are in terms of orbits, their isotropies, and the induced operators. In fact, each of these induced operators, referred to as "a generalized limit operator" above, acts on the product of the corresponding orbit with the corresponding isotropy group and is invariant with respect to that group.

A natural question is to reconcile the classical results on H_N using limit operators with the classical PDEs results based on "generalized limit operators". This is almost done by [70], except that it is not clear whether the resolvents of H_N belong to Vasy's pseudodifferential calculus. (We do know, however, that the resolvents of H_N belong to the norm closure of the pseudodifferential calculus introduced in the previous subsection, as discussed in the previous subsection. Note that here we are using our Theorem 5.11.)

The Fredholm results just mentioned, do apply, however, also to the norm closure of the corresponding pseudodifferential calculi, and hence, in principle, the HVZ theorem could then be obtained from the structure of the orbits of the action of X on $X_{\mathrm{GV}} := [\overline{X} : \mathbb{S}_{\mathcal{F}}]$ and their isotropies (both the orbits and the isotropies are linear subspaces of X) and the explicit form of the generalized limit operators. However, in order not to increase too

much the length of this paper, we leave this for a future publication. Nevertheless, it is interesting to point out that this approach has the potential to provide Fredholm conditions for the restrictions of H_N and its variants to the isotypical components of the action of S_N or some subgroup of S_N. Results in this direction (for operators on compact manifolds) were recently obtained in [9, 8, 10]. See also [19, 22, 35, 60, 61, 63] for related results.

6.4. **A regularity result for bound states.** An application of our results to regularity for bound states for Schrödinger operators with inverse square potentials is contained in our recent preprint [5]. Here we just quickly explain the result. Let

$$(38) \qquad \overline{\mathcal{F}} := \left\{ \overline{Y} \,\middle|\, Y \in \mathcal{F} \right\},$$

which is a clean semilattice that we endow with an admissible order. Let also

$$(39) \qquad X_{\mathcal{F}} := \left[X_{\mathrm{GV}} : \overline{\mathcal{F}} \right] = \left[\overline{X} : \mathbb{S}_{\mathcal{F}} \cup \overline{\mathcal{F}} \right].$$

For instance, if $\mathcal{F} = \left\{ \{0\}, Y_1, Y_2 \right\}$ with $Y_1 \subset Y_2$, then $X_{\mathcal{F}} = \left[\overline{X} : \mathbb{S}_{Y_1}, \mathbb{S}_{Y_2}, \{0\}, \overline{Y}_1, \overline{Y}_2 \right]$. Our results then show that $\left[\overline{X} : \mathbb{S}_{Y_1}, \mathbb{S}_{Y_2}, \{0\}, \overline{Y}_1, \overline{Y}_2 \right] \simeq \left[\overline{X} : \{0\}, \mathbb{S}_{Y_1}, \overline{Y}_1, \mathbb{S}_{Y_2}, \overline{Y}_2 \right]$. See also [48].

For each $Y \in \mathcal{F}$, let $a_Y, b_Y \in \mathcal{C}^\infty(X_{\mathcal{F}})$ and let d_Y denote the distance to Y in some fixed euclidean metric on X. Let also $c \in \mathcal{C}^\infty(X_{\mathcal{F}})$. A function of the form

$$(40) \qquad V(x) := \sum_{Y \in \mathcal{F}} \left(a_Y(x) d_Y(x)^{-2} + b_Y(x) d_Y(x)^{-1} \right) + c(x)$$

will be called an *inverse square potential (associated to the semilattice $\mathcal{F}$)*. Let

$$(41) \qquad \rho(x) := \min \left\{ \operatorname{dist}(x, \textstyle\bigcup \mathcal{F}), 1 \right\},$$

where $\operatorname{dist}(x, \bigcup \mathcal{F})$ is the distance to x to $\mathcal{F}$ in some euclidean metric on X. The following result (which combines techniques of this paper with those in [1]) was proved in [5].

Theorem 6.4. *Let D be a second order strongly elliptic operator with constant coefficients. Let V be an inverse square potential associated to the semilattice $\mathcal{F}$ of linear subspaces of the euclidean space X see Equation (40), $\rho(x) := \min\{\operatorname{dist}(x, \bigcup \mathcal{F}), 1\}$, and assume $u \in L^2(X)$ is an eigenfunction of $D + V$, that is $(D + V)u = \lambda u$ on $X \smallsetminus \bigcup \mathcal{F}$ for some $\lambda \in \mathbb{C}$, then, for all multi-indices α, we have*

$$\rho^{|\alpha|} \partial^\alpha u \in L^2(X) \,.$$

Our theorem covers, of course, the case of the operators H_N and H_{N-1}^{eff} of Equations (1) and (2) (for $D = -\Delta$). Also, we note that, since V is not assumed to be real valued, we do not necessarily have $\lambda \in \mathbb{R}$. The regularity of bound states and, in general, the geometry of the Georgescu–Vasy space $X_{\mathrm{GV}} = [X : \mathbb{S}_{\mathcal{F}}]$ may be useful for approximation purposes. In fact, another, related motivation of our work is the approximation of the isolated eigenfunctions of N-body Hamiltonians using the Finite Element Method. The role of the Georgescu–Vasy space X_{GV} here is to provide a good underlying support for the construction of the approximation spaces. This is very tentative yet, but see [30, 38, 40, 74] for some results in this direction, including more references.

A first motivation for inverse square potentials comes from relativistic physics, where operators of the form "Dirac plus Coulomb potential" are used. The square of these operators will be an operator with inverse square potentials of the type covered by Theorem 6.4. See also [40] and, especially, the recent paper by Dereziński and Richard [27] and the references therein for further physical motivation for inverse square potentials.

APPENDIX A. PROPER MAPS

We now provide a characterization of proper maps used in the main body of the paper. Let $f : X \to Y$ be a continuous map between two Hausdorff spaces. Recall that f is called *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset Y$.

Lemma A.1 (Generalizes [53, Prop 4.32]). *Let $f : X \to Y$ be a continuous map between two Hausdorff spaces with Y locally compact. If f is proper, then f is closed.*

In [53, Prop 4.32] the lemma is stated with the additional requirement that X be locally compact. However in the proof the locally compactness of X is not needed. We omit the proof since we will apply the lemma only when X is locally compact.

Corollary A.2. *Let $f : X \to Y$ be a continuous injective map between two Hausdorff spaces with Y locally compact. If f is proper, then f is a homeomorphism onto its image.*

Proof. The map $f : X \to f(X)$ is bijective continuous and closed and thus a homeomorphism. $\square$

We shall say that f is *locally proper* if, for every $y \in Y$, there exists an open neighborhood V_y of y in Y such that the map $f^{-1}(V_y) \to V_y$ induced by f is proper.

Lemma A.3. *Let $f : X \to Y$ be a continuous map between two Hausdorff spaces with Y locally compact. Then f is proper if, and only if, it is locally proper.*

Proof. Clearly, every proper map is locally proper, by definition. Let us assume that f is locally proper and let $K \subset Y$ be a compact subset. For any $y \in K$ we choose the open neighborhood V_y as in the definition of a locally proper map. As Y is locally compact, there is an open neighborhood W_y of y in V_y such that its closure $\overline{W}_y$ in Y is a compact subset of V_y. The local properness of f together with the choice of V_y implies that $f^{-1}(\overline{W}_y \cap K)$ is compact. By the compactness of K we can choose $y_1, \ldots, y_N$ such that K is covered by $\left(W_{y_j} \right)_{1 \leq j \leq N}$. Then $K = \bigcup_{j=1}^{N} \left(\overline{W}_{y_j} \cap K \right)$. Hence

$$f^{-1}(K) = \bigcup_{j=1}^{N} f^{-1}(\overline{W}_{y_j} \cap K)$$

is also compact. This completes the proof. $\square$

APPENDIX B. MORE ON SUBMANIFOLDS OF MANIFOLDS WITH CORNERS

We discuss here a few other notions of submanifolds and the relation to our concept of weak submanifold. While this is not needed for the proof of the main result, we hope the interested reader will find this material useful.

B.1. **Submanifolds in Melrose's sense.** We begin with Melrose's concept of a submanifold in a manifold with corners, following [56, Definition 1.7.3].

Definition B.1. A subset S of a manifold with corners M of dimension n is a *submanifold (in the sense of manifolds with corners)* if, for every $p \in S$, there exists $0 \leq k \leq n$ and a (corner) chart $\phi : U \to \Omega \subset \mathbb{R}_k^n := [0, \infty)^k \times \mathbb{R}^{n-k}$, numbers $n' \leq n$ and $k' \leq n'$, and a matrix $G \in \mathrm{GL}(n, \mathbb{R})$ such that
  (1) $p \in U$
  (2) $G \left(\mathbb{R}_{k'}^{n'} \times \{0\} \right) \subset \mathbb{R}_k^n$.
  (3) $\phi(S \cap U) = G \left(\mathbb{R}_{k'}^{n'} \times \{0\} \right) \cap \Omega$.

Obviously, every submanifold in the sense of manifolds with corners is a weak sub-manifold, see Definition 2.10. In [56] all submanifolds are submanifolds in the sense of Definition 2.10, see *e. g.,* Lemma 2.11. In Remarks 2.12 (a) we explained that any weak submanifold of a manifold with corners inherits an atlas, and thus this also applies to sub-manifolds in the above sense. However, it can be shown [50] that many submanifolds in our article are not submanifolds in the sense of manifolds with corners, but only weak sub-manifolds, as defined in Definition 2.10. In Example B.3 we provide an example of a weak submanifold of $\mathbb{R}_1^2$ that is not one in the sense of Definition B.1.

Example B.2 (Diagonal)**.** Let N be a manifold with corners. Then $M := N \times N$ is also a manifold with corners. Consider the diagonal $\Delta_N := \{(p, p) \in M \mid p \in N\}$. Then Δ_N is a submanifold of M in the sense of manifolds with corners.

The following provides examples of weak submanifolds that are not submanifolds in the sense of Definition B.1.

Examples B.3.
   (1) The function $f : \mathbb{R}_1^2 := [0, \infty) \times \mathbb{R} \to \mathbb{R}_1^2$, $f(x, y) := (x + y^2, y)$, is an injective immersion. It is a homeomorphism onto its image $S := f\big(\mathbb{R}_1^2\big)$. However, it can be easily seen that S is not a submanifold of $\mathbb{R}_1^2$ in the sense of manifolds with corners. On the other hand S is a submanifold of $\mathbb{R}^2$ in the sense of manifolds with corners.
   (2) The function $f(x) := (x, x^2)$ defines a injective immersion $\mathbb{R}_1^1 \to \mathbb{R}_2^2$. It is a homeomorphism onto its image $S := f\big(\mathbb{R}_1^1\big)$. However, S is not a submanifold $\mathbb{R}_2^2$ in the sense of manifolds with corners. On the other hand S is a submanifold of $\mathbb{R}_1^2$ and of $\mathbb{R}^2$ in the sense of manifolds with corners.

In our article injective immersion which are a homeomorphism to its image, play an im-portant role. Recall the following classical fact for manifolds N and M *without boundary and without corners*:

$$(*) \begin{cases} \text{If } f : N \to M \text{ is an injective immersion, then } f(N) \text{ is a submanifold} \\ \text{if, and only if, } f \text{ maps } N \text{ homeomorphically to } f(N). \end{cases}$$

Examples B.3 show that $(*)$ does no longer hold if M and N are manifolds with corners and if we understand the word "submanifold" in the sense of Definition B.1. On the other, we proved in Proposition 2.13 that $(*)$ holds for manifolds with corners, if we replace "a submanifold" by "a weak submanifold."

B.2. **Other classes of submanifolds.** For comparison and completness, we recall now the definitions of some further classes of submanifolds. The reason the reader might be inter-ested in these concepts is that the concept of a submanifold in the sense of Definition B.1 seems to be too unspecific and the concept of a p-submanifold (Definition 2.15) seems to be sometimes too restrictive. A first alternative is the concept of a "wib-submanifold", where "wib" stands for a submanifold *without an interior boundary.*

Definition B.4. A submanifold $S \subset M$ is called a *wib-submanifold* or a *submanifold without interior boundary* if it can be defined locally in suitable charts as the kernel of a linear function. More precisely: $S \subset M$ is a *wib-submanifold* if, for every $x \in S$, there exists a (corner) chart $\phi : U \to \Omega \subset \mathbb{R}_k^n$, and a linear subspace L of $\mathbb{R}^n$, such that
   (1) $x \in U$ and
   (2) $\phi(S \cap U) = L \cap \Omega$.

If $G \in \mathrm{GL}(n, \mathbb{R})$ is as in Definition B.1, then we necessarily have $L = G\left(\mathbb{R}^{n'} \times \{0\}\right)$. If $x \in S \cap U$, then $n' := \dim(L)$ is the dimension of S in x defined above. Obviously all p-submanifolds are wib-manifolds, which can be easily seen by defining the L in the definition above as the linear extension of L_I in Definition 2.15.

Remark B.5. In the above definition, we explicitly required S to be a submanifold. To justify this requirement, we will give an example of a closed subset $S \subset M$ that is not a submanifold, but fulfills all other requirements of the definition of a wib-submanifold. Indeed, let

$$K := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \geq 0, \ x_2 \geq 0, \ x_1 \leq x_3, \ x_2 \leq x_3 \right\},$$

which is a cone over a square. The map $f : \mathbb{R}^3 \to \mathbb{R}^4$, $f(x_1, x_2, x_3) = (x_1, x_2, x_3 - x_1, x_3 - x_2)$ has the property $f^{-1}(\mathbb{R}_4^4) = K$. Then for $\phi = \mathrm{id}$, $x = 0$, and $L := f(\mathbb{R}^3)$ all requirements of the definition are satisfied, but $S := f(K)$ is not a submanifold of $\mathbb{R}_4^4$. It it were a submanifold, then its dimension would have to be 3, and then any boundary point of S is in at most 3 closed boundary hyperfaces. But $0 \in S$ is in 4 closed boundary hyperfaces of S.

Remark B.6. Note that Melrose also introduces the notions d-submanifold [56, Def. 1.7.4] and b-submanifold [56, Def. 1.12.9], whose definitions will not be recalled here. They satisfy

$$S \text{ is a p-submanifold} \implies S \text{ is a d-submanifold} \implies S \text{ is a b-submanifold}$$

$$\implies S \text{ is a submanifold} \implies S \text{ is a weak submanifold}.$$

However there are wib-manifolds that are not b-submanifolds, such as Melrose's example of the submanifold $\{x_3 = x_1 + x_2\} \in \mathbb{R}_3^3$. There are d-manifolds that are no wib-manifolds, for instance, $\mathbb{R}_1^1 = [0, \infty) \subset \mathbb{R}$ or any surface with boundary in $\mathbb{R}^3$. However all p-submanifolds introduced below are d-submanifolds and wib-submanifolds. Melrose shows that the diagonal Δ_N is a b-submanifold of $N \times N$, but in general not a d-submanifold. It follows that Δ_N is not a p-submanifold.

Remark B.7. Let us remark that the concept of a *tame* submanifold considered in [2, Sec. 2.3] is a concept of a submanifold in an essentially different sense, it is actually a more restrictive notion of submanifold than the ones encountered in this paper. All notions of submanifolds discussed so far involve properties that may or may not hold for a subset N of a manifold with corners M. In contrast to this, tame submanifolds in [2, Sec. 2.3] are submanifolds of a *Lie manifold* (M, A), where M is a manifold with corners and A is a Lie algebroid on M with some compatibility conditions. Whether a subset N of M is a tame submanifold of (M, A) or not depends also on the Lie algebroid A. In any case, a tame submanifold will have a tubular neighborhood in the strongest sense. Similar remarks apply to the $A(\mathcal{G})$-tame submanifolds considered in [66].

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