Nonlinear eigenvalue problems on Riemannian manifolds

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Non-linear eigenvalue problems on domains in $\mathbb{R}^n$

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $n \geq 3$.
We are interested in solutions
$u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of

$$\begin{aligned} \Delta u &= c|u|^{p-2}u && \text{auf } \Omega \\ u &= 0 && \text{auf } \partial\Omega \end{aligned} \tag{1}$$

Notation:

$$\Delta u = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u$$

$$C_0^2(\Omega) = \left\{ u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \,\middle|\, u|_{\partial\Omega} = 0 \right\}.$$

Content of the talk

I. Non-linear eigenvalue problems on domains in $\mathbb{R}^n$

1. The linear case
2. The subcritical case $2 < p < 2n/(n-2)$
3. Results for the critical case $p = 2n/(n-2)$
4. Perturbation approach to the critical equation
5. Example
6. Geometrisation
7. Construction of a blow-up limit

II. The Yamabe-Problem

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III. Proof(s) of the Positive Mass Theorem

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1. The linear case, i.e. $p = 2$

The linear case $p = 2$ is very well understood:
There are real numbers

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$$

and functions $v_i \in C_0^2(\Omega) \setminus \{0\}$, such that

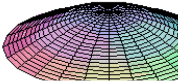
$$\Delta v_i = \lambda_i v_i.$$

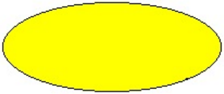
The functions v_i are pairwise orthogonal. The vector space spanned by the vectors v_i is dense in $C_0^2(\Omega)$.

$$v_1(x) > 0 \qquad \forall x \in \Omega.$$

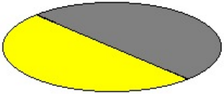
Eigenfunctions on a disc in $\mathbb{R}^2$ in the linear case

λ_1

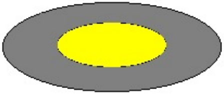




$\lambda_2 \approx 1.59\lambda_1$



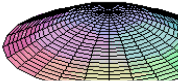
$\lambda_4 \approx 2.30\lambda_1$

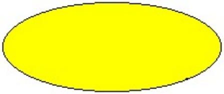


Quelle: www.chemgapedia.de

Eigenfunctions correspond to vibrating membranes

λ_1





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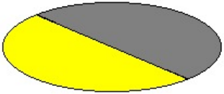

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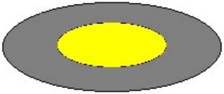

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2. The subcritical case $2 \leq p < 2n/(n-2)$

Many Methods from the linear case $p = 2$ still work.

Proposition

For $2 \leq p < 2n/(n-2)$ *there is a solution of* (1)*, that is positive and smooth on* Ω*.*

Idea of proof:

$$\mathcal{F}_p(u) := \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^p\right)^{\frac{2}{p}}},$$

The Sobolev embedding

$$H^{1,2}(\Omega) \to L^p(\Omega)$$

is compact. This implies

$$\mu_p(\Omega) := \inf_{u \in C_0^2(\Omega)} \mathcal{F}_p(u) > 0.$$

Now choose a sequence $u_i \in C_0^2(\Omega)$ such that

$$\int_\Omega |u_i|^p = 1, \qquad \int_\Omega |\nabla u_i|^2 \to \mu_p(\Omega)$$

For a subsequence of (u_i) we get

$$u_i \to u_\infty$$

weakly in $H^{1,2}(\Omega)$ and strongly in $L^p(\Omega)$.
The function u_∞ is a minimizer for $\mathcal{F}_p$ and satisfies the associated Euler-Lagrange equation

$$\begin{aligned} \Delta u_\infty &= \mu_p(\Omega)\, |u_\infty|^{p-2} u_\infty, \\ u_\infty &> 0 \qquad\qquad \text{auf } \Omega, \\ u_\infty &\in C_0^2(\Omega), \\ \int |u_\infty|^p &= 1. \end{aligned}$$

3. Results for the critical case $p = 2n/(n-2)$

More involved.
In this case equation (1) is called the Nirenberg equation.

- The previous existence proof fails, as the Sobolev embedding $H^{1,2}(\Omega) \to L^p(\Omega)$ is no longer compact. However, it is still bounded.
- Is there still a positive and smooth solution?
  - There is never a minimizing solution.
  - There is no solution at all if Ω is star-shaped.
  - However, if $H_d(\Omega, \frac{\mathbb{Z}}{2\mathbb{Z}}) \neq 0$ for some $d > 0$, then there is a positive smooth solution.

4. Perturbation approach to the critical equation

Idea: For $p < 2n/(n-2)$ we determine a solution of

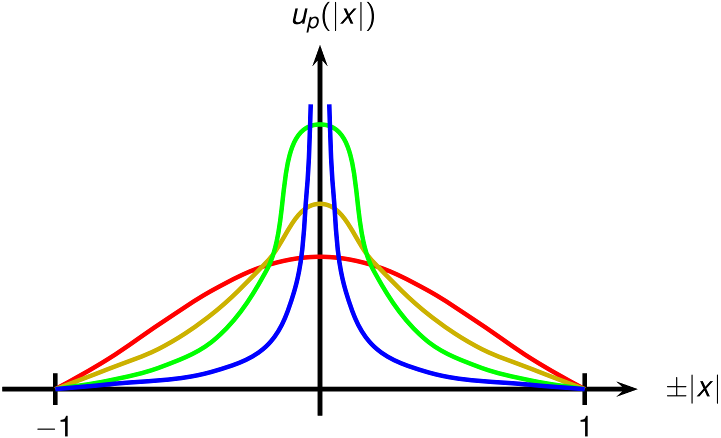
$$\begin{aligned} \Delta u_p &= \mu_p \, |u_p|^{p-2} u_p && \text{on } \Omega \\ u_p &> 0 && \text{on } \Omega \\ u_p &\in C_0^2(\Omega) \\ \textstyle\int |u_p|^p &= 1 \\ \mu_p &= \inf \mathcal{F}_p \end{aligned}$$

Is there a limit $u_p \to u_{\text{crit}}$ for $p \to p_{\text{crit}} = 2n/(n-2)$?

Fact: $[2, p_{\text{crit}}] \ni p \mapsto \mu_p$ is continuous.

5. Example

$\Omega =$ Unit ball in $\mathbb{R}^n$, $u_p(x) = u_p(|x|)$.
The functions u_p are radially symmetric.



$p = 2$ $p = 4$ $p = 5$ $p = 5,99$

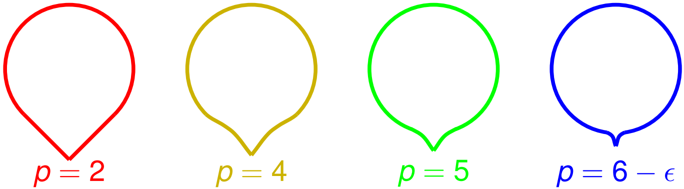
$n = 3$

${u_p}^{\frac{2n}{n-2}}\, d\mu \to \delta_0$ weakly in the sense of measures.
$u_p \to 0$ weakly in $L^{\frac{2n}{n-2}}(\Omega)$.

6. Geometrisation

$\Omega =$ unit ball in $\mathbb{R}^n$ with euclidean metric g_{eucl}

For $p < \frac{2n}{n-2}$ we define the Riemannian metric $\tilde{g}_p := u_p^{\frac{4}{n-2}} g_{\mathrm{eucl}}$.
Drawings for $(\Omega, \tilde{g}_p)$:



Convergence to a round sphere
There is a "‘Blow-up"’ to a round sphere

7. Construction of a blow-up limit

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II. The Yamabe-Problem

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III. Proof(s) of the Positive Mass Theorem

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