

# Nonlinear eigenvalue problems on Riemannian manifolds

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November 30, 2017

## Overview of my talk in the Ringvorlesung

### Thursday, November 23, 2017

The talk will start with the discussion of the following problem. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with smooth boundary. For  $p \in [2, \infty)$  we consider the equation

$$\begin{aligned} \Delta u &= \lambda |u|^{p-2} u && \text{on } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1}$$

The case  $p = 2$  is the classical linear case, which is very well understood. Many results from the linear case can be generalized to the case of a *subcritical* non-linearity, which is by definition the case  $2 < p < 2n/(n-2)$ . The analysis changes essentially for  $p = 2n/(n-2)$ , the case of a *critical* non-linearity. The reason for the different behavior is that in the critical case equation (1) is “scaling invariant”, it is in fact a special case of a conformally invariant equation in Riemannian geometry, the so-called *Yamabe equation*. In the first part of the talk we will discuss some existence and non-existence result for equation (1). In particular, we will interpret solutions of (1) as critical points of a functional, and we will show, that on domains  $\Omega$  as above there are no solutions of (1) that minimize this functional.

The second part of the talk will discuss the Yamabe problem. We assume that  $M$  is a manifold with a Riemannian metric  $g$ . To such a metric one can associate its scalar curvature which is a function  $\text{scal}^g : M \rightarrow \mathbb{R}$ . The Yamabe problem asks: is it possible to change  $g$  in a conformal way to a metric  $\tilde{g} := fg$ ,  $f > 0$  smooth, such the scalar curvature  $\text{scal}^{\tilde{g}}$  for this new metric is constant? To discuss this problem, it is helpful to rewrite this condition in terms of  $u$ , where  $f = u^{4/(n-2)}$ . Then the condition that  $\text{scal}^{\tilde{g}}$  is equal to the constant  $s_0 = \frac{4(n-1)}{n-2} \lambda$  is equivalent to

$$\left( \Delta + \frac{n-2}{4(n-1)} \text{scal}^g \right) u = \lambda |u|^{p-2} u. \tag{2}$$

Apparently this equation generalizes (1). For simplicity we now assume that  $M$  has no boundary. We will explain, why the Yamabe equation admits a solution minimizing the associated functional on every compact manifold  $M$  (without boundary). The problem requires a proof of the positive mass theorem in general relativity which was proven by Schoen and Yau. This theorem will be one of the topics of my second talk.

## Outlook

Let me spend some few words, why this special kind of non-linear equation is so important in geometric analysis, although we will probably not discuss this aspect in the talk.

Such equations appear naturally in applications in conformal geometry, in general relativity and geometric topology, typically in equations which are invariant under rescaling (in some sense). Consider for example the Einstein equations in general relativity which describe the evolution our space-time and thus also gravitational waves. These equations are invariant under the diffeomorphism group, which in particular allows to expand regions of the manifolds which is such a kind of rescaling. Another example is the Ricci-flow which is used in the Hamilton-Perelman program to prove the Poincaré conjecture.

One can show that the diffeomorphism invariance implies that the Einstein equations are not elliptic on a Riemannian manifold, that the Einstein equations are not hyperbolic on a Lorentzian manifold and that the Ricci flow is not parabolic.

The non-linearities in the equations discussed in our lecture are much simpler than these diffeomorphism invariant equations. They are invariant under conformal changes of the metric, which includes invariance under angle preserving diffeomorphisms. However, they still show some key features, and thus they have the role of important models.

## About this file

File compiled: November 30, 2017

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