Nonlinear eigenvalue problems on Riemannian manifolds — Handout

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Abstract

This is a (partial) handout from a Ringvorlesung of the Graduiertenkolleg GRK 1692 in mathematics in Regensburg, Germany, delivered by Bernd Ammann in November 2017.

For $p \in [2, 2n/(n-2)]$, $n \geq 3$ we discuss the existence of solutions the following equation on an n-dimensional bounded domain Ω

$$\begin{aligned} \Delta u &= \lambda |u|^{p-2} u \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}.$$

Afterwards we turn our attention to the Yamabe problem on compact manifolds. We sketch how the existence of a solution of a solution can be derived with the help of the positive mass theorem in general relativity. The last part of the lecture discusses different approaches to proof this theorem under varying assumptions.

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I Non-linear eigenvalue problems on bounded domains in $\mathbb{R}^n$

The following part will be explained in the slides, available at this link. This is why the following part is in the form of an extended abstract.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. For $p \in [2, \infty)$ we consider the equation

$$\begin{aligned} \Delta u &= \lambda |u|^{p-2} u \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

The case $p = 2$ is the classical linear case, which is very well understood. Many results from the linear case can be generalized to the case of a *subcritical* non-linearity, which is by definition the case $2 < p < 2n/(n-2)$. The analysis changes essentially for $p = 2n/(n-2)$, the case of a *critical* non-linearity. The reason for the different behavior is that in the critical case equation (1) is "scaling invariant", it is in fact a special case of a conformally invariant equation in Riemannian geometry, the so-called *Yamabe equation.* In the first part of the lecture we will discuss some existence and non-existence result for equation (1). In particular, we will interpret solutions of (1) as critical points of a functional, and we will show, that on domains Ω as above there are no solutions of (1) that minimize this functional.

Side remark about the context

Let me spend some few words, why this special kind of non-linear equation is so important in geometric analysis, although we will probably not discuss this aspect in the lecture.

Such equations appear naturally in applications in conformal geometry, in general relativity and geometric topology, typically in equations which are invariant under rescaling (in some sense). Consider for example the Einstein equations in general relativity which describe the evolution our space-time and thus also gravitational waves. These equations are invariant under the diffeomorphism group, which in particular allows to expand regions of the manifolds which is such a kind of rescaling. Another example is the Ricci-flow which is used in the Hamilton-Perelman program to prove the Poincaré conjecture.

One can show that the diffeomorphism invariance implies that the Einstein equations are not elliptic on a Riemannian manifold, that the Einstein equations are not hyperbolic on a Lorentzian manifold and that the Ricci flow is not parabolic.

The non-linearities in the equations discussed in our lecture are much simpler than these diffeomorphism invariant equations. They are invariant under conformal changes of the metric, which includes invariance under angle preserving diffeomorphisms. However, they still show some key features, and thus they have the role of important models.

7 Construction of a blow-up limit

Here the presentation via video projector ends.

Our goal now is to turn the geometric construction that was indicated on the last slides (Subsection I.6) into a precise argument.

In this subsection we assume that Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. For a sequence of $p_i \to p_{\text{crit}} = 2n/(n-2)$, we assume that we have solutions $u_i \in C_0^2(\Omega)$

$$\begin{aligned} \Delta u_i &= \lambda_i |u_i|^{p_i-2} u_i && \text{on } \Omega \\ u_i &= 0 && \text{on } \partial\Omega \\ u_i &> 0 && \text{on } \Omega \\ \textstyle\int_\Omega u_i^{p_i} &= 1 \end{aligned}$$

Example I.1. *Examples are the minimizing solutions discussed before.*

We choose $x_i \in \Omega$ with $u_i(x_i) = m_i := \max u_i$.

Proposition I.2. *Assume that Ω, p_i, λ_i, u_i, x_i, m_i are as above. Furthermore assume that $d(x_i, \partial\Omega) \geq c_0 > 0$, $\lambda_i \to \lambda_\infty$ and $m_i \to \infty$. Then*

$$\tilde{u}_i(x) := \frac{1}{m_i} u_i(m_i^{-(p_i-2)/2}(x - x_i))$$

solves the same non-linear equation, i.e.

$$\begin{aligned} \Delta \tilde{u}_i &= \lambda_i |\tilde{u}_i|^{p_i-2} \tilde{u}_i && \textit{on } \Omega_i \\ \tilde{u}_i &= 0 && \textit{on } \partial\Omega_i \\ \tilde{u}_i &> 0 && \textit{on } \Omega_i \\ \textstyle\int_{\Omega_i} \tilde{u}_i^{p_i} &= 1 \end{aligned}$$

where $\Omega_i := \{x \in \mathbb{R}^n \mid m_i^{-(p_i-2)/2}(x - x_i) \in \Omega\}$.

Proof. This was proven in the lecture. $\square$

Note that $\bigcup_i \Omega_i = \mathbb{R}^n$.

Proposition I.3. *$\tilde{u}_i$ converges to a function $u_\infty : \mathbb{R}^n \to \mathbb{R}$ in $C^2(B_R(0))$ for each $R > 0$.*

Proof. The statement follows from regularity theory using a boot-strap argument, which was only sketched in the lecture. $\square$

We have thus obtained a solution of

$$\begin{aligned} \Delta u_\infty &= \lambda_\infty |u_\infty|^{p_{\mathrm{crit}}-2} u_\infty && \text{on } \mathbb{R}^n \\ u_\infty &> 0 && \text{on } \mathbb{R}^n \\ 0 < \textstyle\int_{\mathbb{R}^n} u_\infty^{p_{\mathrm{crit}}} &\leq 1 \end{aligned}$$

We will see later that this will imply that $(\mathbb{R}^n, u_\infty^{4/(n-2)} g_{\mathrm{eucl}})$ is isometric to a round sphere[1] of volume $\int_{\mathbb{R}^n} u_\infty^{p_{\mathrm{crit}}}$ with one point removed. This implies

$$\lambda_\infty \left(\int_{\mathbb{R}^n} u_\infty^{p_{\mathrm{crit}}} \right)^{2/n} = n(n-1)(\omega_n)^{2/n}$$

where ω_n is the volume of the round sphere of radius 1, i.e. the standard sphere.

II The Yamabe Problem

Literature

For an introduction to differential geometry we recommend the C. Bär's German lecture notes [1], which are also available in an English version.

For the following parts the overview article [3] is a good reference for many aspects. Several foundations are presented in more detail in other lecture notes by C. Bär [2]. Unfortunately, Lee and Parker's article [3] contains a gap at the end of Proposition 4.6 which is very well hidden, and there is probably no way to repair it without reorganizing Sections 3 and 4 in that article. The problem still affects the corresponding part in Bär's lecture notes [2] in its current version (Fall 2017).

The lecture will present a solution of the Yamabe problem which fills this gap, based on blowup techniques which I learnt from lecture notes by Schoen.

1 Introduction

We assume that M is a compact manifold with a Riemannian metric g. For simplicity we only consider manifolds without boundary. To any Riemannian metric one can associate its scalar curvature which is a function $\mathrm{scal}^g : M \to \mathbb{R}$. The Yamabe problem asks: is it possible to change g in a conformal way to a metric $\tilde{g} := fg$, $f > 0$ smooth, such the scalar curvature $\mathrm{scal}^{\tilde{g}}$ for this new metric is constant? To discuss this problem, it is helpful to rewrite this condition in

[1] A round n-dimensional sphere is by definition a Riemannian manifold isometric to $\{x \in \mathbb{R}^{n+1} \mid \|x\| = r\}$ for some Radius r.

terms of u, where $f = u^{4/(n-2)}$. Then the condition that $\mathrm{scal}^{\tilde{g}}$ is equal to the constant $s_0 = \lambda$ is equivalent to

$$\left(\frac{4(n-1)}{n-2} \Delta + \mathrm{scal}^g \right) u = \lambda |u|^{p-2} u. \tag{2}$$

Apparently this equation would generalizes (1) if we would extend it to manifolds with boundary assuming Dirichlet boundary conditions. So this equation is tigthly related to the equation in the first section.

In the lecture we explain, why the Yamabe equation admits a solution minimising the associated functional on every compact manifold M (without boundary).

2 Yamabe's dream

3 Perturbation approach

4 Using conformal coordinates to prove the Yamabe inequality

In this subsection we will sketch how to prove the Yamabe inequality under the additional assumptions that $n = \dim M \geq 6$ and that the (M, g_0) is not conformal to the round sphere.

5 Proving the Yamabe inequality with the Positive Mass Theorem

In this subsection we will sketch how to prove the Yamabe inequality ion the remaining cases, i.e. we assume either that $n = \dim M \in \{3, 4, 5\}$ or that (M, g_0) is conformal to the round sphere.

III Proof(s) of the Positive Mass Theorem

The problem requires a proof of the positive mass theorem in general relativity which was proven by Schoen and Yau in the relevant cases.

References

[1] Bär, C. Differentialgeometrie. Vorlesungsskript (unveröffentlicht), 2006. erhältlich unter http://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-Geometrie/Dokumente/Lehre/Lehrmaterialien/skript-DiffGeoErw.pdf.

[2] Bär, C. Geometrische Analysis. Vorlesungsskript (unveröffentlicht), 2007/08. erhältlich unter http://www.math.uni-potsdam.de/fileadmin/user_upload/Prof-Geometrie/Dokumente/Lehre/Lehrmaterialien/skript-GeomAna.pdf.

[3] Lee, J. M., and Parker, T. H. The Yamabe problem. *Bull. Am. Math. Soc., New Ser. 17* (1987), 37–91.

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