Parallel spinors on Riemannian and Lorentzian manifolds

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Images/150JapanGermany.jpg

Talk I: Parallel spinors on Riemannian manifolds

joint work with Klaus Kröncke, Hartmut Weiß, and Frederik Witt
Arxiv 1512.07390, Web-Link with further info
Math. Z. **291** *303-311* (2019)

A) Basic facts about Ricci-flat manifolds, special holonomy and spin geometry
B) Our results
C) Applications to the space of metrics g with $\mathrm{scal}^g \geq 0$
(D. Wraith, T. Schick)

Images/zahnrad-germany-and-japan.jpg

Talk II:

A) Parallel spinors on Lorentzian manifolds

(joint work with Klaus Kröncke and Olaf Müller)
Arxiv 1903.02064

B) Topology of initial data sets with strict dominant energy condition

(work by Jonathan Glöckle)
Arxiv 1906.00099

Images/zahnrad-germany-and-japan.jpg

Goal for Talk I

Study the space of all Ricci-flat metrics on a given compact manifold M.

Main interest:

(Pre-)Moduli space $\{\text{Ricci-flat metrics}\}/\,\mathrm{Diff}_{\mathrm{Id}}(M)$

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C) Applications to the space of metrics g with $\mathrm{scal}^g \geq 0$
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Cheeger-Gromoll splitting theorem

Theorem (Cheeger-Gromoll splitting theorem)

Let (M, g) *be a complete Riemannian manifold with* $\mathrm{Ric}^M \geq 0$*.*
Suppose $\gamma : \mathbb{R} \to M$ *is a line, i.e. a geodesic satisfying*

$$\forall t, s \in \mathbb{R} : \; d(\gamma(t), \gamma(s)) = |t - s| \, .$$

Then we have

$$(M, g) = (M_0, g_0) \times \mathbb{R} \, .$$

My favorite reference:
Takashi Sakai, Riemannian Geometry, AMS, Theorem V.3.9
Following work by J. Eschenburg and E. Heintze (Augsburg in Bavaria within Germany)

Images/zahnrad-germany-and-japan

Consequence of Cheeger-Gromoll splitting theorem

Lemma

If (M, g) is a complete Riemannian manifold with infinite diameter. Let Γ act cocompactly and isometrically on (M, g). Then (M, g) carries a line.

The Cheeger-Gromoll splitting theorem implies:

Theorem

If (M, g) is a compact Riemannian manifold with $\mathrm{Ric}^M \geq 0$. Then the universal covering $\widetilde{M}$ is isometric to

$$N \times \mathbb{R}^k$$

where N is a simply-connected compact manifold with $\mathrm{Ric}^N \geq 0$.

Note that k is a topological invariant, namely the growth rate of $\pi_1(M)$.

Non-positive Ricci curvature

Lemma

Let K be a Killing vector field on a Riemannian manifold, then

$$\nabla^*\nabla K = \mathrm{Ric}(K)$$

Proof

Let K be a Killing field on a Riemannian manifold. Then $Y \mapsto \nabla_Y K$ is skew-symmetric, and thus $Y \mapsto \nabla^2_{X,Y} K$ is also skew-symmetric for fixed X.

$$\begin{aligned}
0 &= g(R(X,Y)K,W) + g(R(Y,W)K,X) + g(R(W,X)K,Y) \\
&= g(\nabla^2_{X,Y}K,W) - g(\nabla^2_{Y,X}K,W) + g(R(Y,W)K,X) \\
&\quad + g(\nabla^2_{W,X}K,Y) - g(\nabla^2_{X,W}K,Y) \\
&= 2g(\nabla^2_{X,Y}K,W) + 2g(R(Y,W)K,X)
\end{aligned}$$

Thus

$$\nabla^2_{X,Y}K + R(K,X)Y = 0.$$

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Taking the trace (= metric of X- and Y-component) we get

$$\nabla^*\nabla K = \mathrm{Ric}(K)$$

which is the lemma. □

Corollary

If (N,g) is a compact Riemannian manifold with $\mathrm{Ric} \leq 0$, then every Killing vector field is parallel. If such an N is simply connected then its isometry group is finite.

Proof.

$$\int_N \|\nabla K\|^2 = \int_N \langle \mathrm{Ric}(K), K\rangle \leq 0.$$

□

Thus: $\mathrm{Ric}^N \leq 0$ and $\pi_1(N) = \{1\}$ implies $\# \mathrm{Isom}(N) < \infty$.

Ricci-flat manifolds

Combining $\mathrm{Ric} \geq 0$ and $\mathrm{Ric} \leq 0$ one gets:

Corollary

If (M, g) is a compact Riemannian manifold with $\mathrm{Ric} = 0$*.*
Then $\widetilde{M}$ is isometric to

$$N \times \mathbb{R}^k$$

where N is a simply-connected compact manifold with $\mathrm{Ric}^N = 0$*,* **and** $\mathrm{Isom}(N)$ **is finite***.*

By carefully studying the action of the Deck transformations one obtains: (Web-Link)

Theorem (Cheeger-Gromoll, Fischer-Wolf)

Every (closed) Ricci-flat manifold has a finite normal covering of the form

$$N \times (\mathbb{R}^k/\Gamma)$$

where Γ is a lattice in $\mathbb{R}^k$, and where N is a closed Riemannian manifold with $\pi_1(N) = \{1\}$.

Holonomy

Definition

The holonomy group

$\mathrm{Hol}(M, g) := \{P_\gamma \mid \gamma \text{ path from } p \text{ to } p \text{ in } M\} \subset \mathrm{GL}(T_pM).$

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The restricted holonomy group

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Theorem (de Rham splitting theorem)

Let $(\widetilde{M},\tilde{g})$ be a complete simply-connected Riemannian manifold. Then as a Riemannian product

$$\widetilde{M} = M_1 \times \cdots \times M_r \times \mathbb{R}^k$$

where each M_i is a complete non-flat irreducible Riemannian manifold.

For $\mathrm{Ric} \geq 0$ on a univ. covering $\widetilde{M}$ of a closed M this refines the Cheeger-Gromoll splitting.

Berger's holonomy list

Theorem

If (M, g) *is an irreducible Riemannian manifold,* $n = \dim M$*, then* (M, g) *is locally symmetric or* $\mathrm{Hol}_0(M, g)$ *is one of the following:*

(1) $\mathrm{Hol}_0(M, g) = \mathrm{SO}(n)$ *(generic)*
(2) $\mathrm{Hol}_0(M, g) = \mathrm{U}(n/2)$ *(Kähler)*
(3) $\mathrm{Hol}_0(M, g) = \mathrm{SU}(n/2)$ *(Ricci-flat Kähler= Calabi-Yau)*
(4) $\mathrm{Hol}_0(M, g) = \mathrm{Sp}(n/4)$ *(hyper-Kähler)*
(5) $\mathrm{Hol}_0(M, g) = \mathrm{Sp}(n/4) \cdot \mathrm{Sp}(1)$ *(quaternionic-Kähler)*
(6) $\mathrm{Hol}_0(M, g) = \mathrm{G}_2$ *and* $n = 7$
(7) $\mathrm{Hol}_0(M, g) = \mathrm{Spin}(7)$ *and* $n = 8$

There are compact examples in each case.
In cases (3), (4), (6) and (7) we have $\mathrm{Ric} = 0$.
If M is compact with $\mathrm{Ric} = 0$, then we are in case (1), (3), (4), (6) or (7).
The locally symmetric factors are Ricci flat only if they are flat.

Summary

Let (M, g) be a compact Ricci-flat manifold.
After passing to a finite covering $\widehat{M} \to M$, we can decompose

$$\widehat{M} = M_1 \times \cdots \times M_r \times T^k,$$

where all M_i have irreducible holonomy.

Spin geometry

Let (N, h) be a time- and space-oriented semi-Riemannian manifold.
We assume that we have a fixed spin structure, i.e. a choice of a complex vector bundle $\$^h N$, called the spinor bundle, with

$$\$^h N \otimes_{\mathbb{C}} \$^h N = \bigwedge^{\bullet/\text{even}} T^* N \otimes_{\mathbb{R}} \mathbb{C}.$$

This bundle carries (fiberwise over $p \in M$)

- a non-degenerate hermitian product (positive definit in the Riemannian case)
- a compatible connection
- a compatible Clifford multiplication $\mathrm{cl} : TN \otimes \$^h N \to \$^h N$, $\mathrm{cl}(X \otimes \varphi) =: X \cdot \varphi$ such that

$$X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi + 2h(X, Y)\, \varphi = 0.$$

Spinors and holonomy

Let (N, h) be a Riemannian or Lorentzian spin manifold. Parallel transport along a loop $c : [0, 1] \to N$, $p = c(0) = c(1)$ gives a map

$$\begin{array}{ccc} \mathcal{P}^{\$^h N}(c) \in & \mathrm{Spin}(\$^h_p N) \subset \mathrm{U}(\$^h_p N) \\ & \downarrow 2:1 \\ \mathcal{P}^{TN}(c) \in & \mathrm{SO}(T_p N, h) \end{array}$$

Assume that $\varphi \neq 0$ is a parallel spinor, i.e. a parallel section of $\Gamma(\$^h N)$. Then the holonomy group

$$\mathrm{Hol}(N, h, p) := \{\mathcal{P}^{TN}(c) \mid c \text{ loop based in } p\} \subsetneq \mathrm{SO}(T_p N, h).$$

is special, i.e. $\dim \mathrm{Hol}(N, h, p) < \dim \mathrm{SO}(T_p N, h)$.

Parallel spinors

Let (N, h) be a Riemannian or Lorentzian spin manifold.
Assume that $\varphi \neq 0$ is a parallel spinor,

$\Rightarrow \quad R_{X,Y}\varphi = 0$

$\Rightarrow \quad 0 = \sum \pm e_i \cdot R_{e_i,Y}\varphi \stackrel{!}{=} -\frac{1}{2}\operatorname{Ric}(Y) \cdot \varphi$

$\Rightarrow \quad h(\operatorname{Ric}(Y), \operatorname{Ric}(Y))\varphi = -\operatorname{Ric}(Y) \cdot \operatorname{Ric}(Y) \cdot \varphi = 0$

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In the Riemannian case: $\operatorname{Ric} = 0$

In the Lorentzian case:

$\quad$ $\operatorname{Ric}(Y)$ is lightlike for all Y

$\Rightarrow \quad \operatorname{Ric} = f\alpha \otimes \alpha$ for a lightlike 1-form α.

Remark

A product $M_1 \times \cdots \times M_k$ of (semi-)Riemannian spin manifolds carries a parallel spinor if and only if each factor carries a parallel spinor.

Structured Ricci-flat metrics

$\mathcal{M}(M) := \{\text{Riemannian metrics } g \text{ on } M\}$
$\mathcal{M}_0(M) := \{g \in \mathcal{M}(M) \mid \mathrm{Ric}^g = 0\}.$

Definition

A Riemannian metric g on a compact manifold M is called structured if

- the universal covering $\widetilde{M}$ is spin and
- if $\widetilde{M}$ carries a non-zero parallel spinor.

$\mathcal{M}_{||}(M) := \{g \in \mathcal{M}(M) \mid g \text{ structured}\} \subset \mathcal{M}_0(M)$

$\mathcal{M}_{||}(M)$ is open and closed in $\mathcal{M}_0(M)$. The word "structured Ricci-flat": there is a frist order structure which implies Ricci-flatness.

Open Problem

Is every Ricci-flat metric on a compact manifold structured?
$\mathcal{M}_{||}(M) \subsetneq \mathcal{M}_0(M)$?

Berger's list, refined

If (M, g) is an irreducible Riemannian manifold, $n = \dim M$, then (M, g) is locally symmetric or $\mathrm{Hol}_0(M, g)$ is one of the following:

$\mathrm{Hol}_0(M, g)$	Name	structured	$\mathrm{Ric} = 0$
$\mathrm{SO}(n)$	generic	No	??
$\mathrm{U}(n/2)$	Kähler	No	No
$\mathrm{SU}(n/2)$	Calabi-Yau	Yes	Yes
$\mathrm{Sp}(n/4)$	Hyper-Kähler	Yes	Yes
$\mathrm{Sp}(n/4) \cdot \mathrm{Sp}(1)$	quatern.-Kähler	No	No
G_2, $n = 7$	G_2	Yes	Yes
$\mathrm{Spin}(7)$, $n = 8$	$\mathrm{Spin}(7)$	Yes	Yes

?? = "maybe in some cases"

Linearization of $\mathrm{Ric}^g = 0$ and stability

Einstein-Hilbert functional $\mathcal{E}(g) = \int_M \mathrm{scal}^g \, dv^g$.
Critical points = Ricci-flat metrics:

$$d\mathcal{E}|_g = 0 \;\Leftrightarrow\; \mathrm{Ric}^g = 0.$$

This holds for all semi-Riemannian metrics.
Ricci-flat Lorentzian metrics are called vacuum solutions of the Einstein equations.

Images/EinsteinRicNull136.pdf

Linearization of $\mathrm{Ric}^g = 0$ and stability (Norihito Koiso)

Einstein-Hilbert functional $\mathcal{E}(g) = \int_M \mathrm{scal}^g \, dv^g$.

Critical points = Ricci-flat metrics.

One can consider deformations

- inside a conformal class
  $\rightsquigarrow$ no new Ricci-flat metrics except by rescaling
- by diffeomorphisms
  $\rightsquigarrow$ geometry unchanged
- orthogonal to conformal classes and diffeomorphism orbits
  "interesting directions"
  $\rightsquigarrow$ TT-tensors

$\mathrm{TT}^+ := \{h \text{ symm. } (0,2)\text{-tensor} \mid \mathrm{tr}_g h = 0 \text{ and } \mathrm{div}_g h = 0\} \oplus \mathbb{R}g.$

For $h, k \in \mathrm{TT}^+$, $\mathrm{Ric}^g = 0$ we have

$$(\mathrm{Hess}_g \, \mathcal{E})(h,k) = -\frac{1}{2} \int_M \langle h, \Delta_E k \rangle \; \mathrm{dvol}^g$$

where $\Delta_E = \nabla^* \nabla - 2\mathring{R}$ is the Einstein operator.

On Lorentzian manifolds elements of $\mathrm{TT} \cap \ker \Delta_E$ are the gravitational waves.

Notation

$\mathcal{M}(M) := \{\text{Riemannian metrics } g \text{ on } M\}$
$\mathcal{M}_0(M) := \{g \in \mathcal{M}(M) \mid \mathrm{Ric}^g = 0\}$
$\mathrm{Diff}(M) := \{\text{Diffeomorphisms } M \to M\}$
$\mathrm{Diff}_{\mathrm{Id}}(M)$ is the identity component of $\mathrm{Diff}(M)$.

$\mathrm{Diff}(M)$ acts on $\mathcal{M}(M)$ and on $\mathcal{M}_0(M)$ via pullback

Main interest of Talk I A&B

Moduli space $\mathcal{M}_0(M)/\,\mathrm{Diff}(M)$,
Premoduli space $\mathcal{M}_0(M)/\,\mathrm{Diff}_{\mathrm{Id}}(M)$

Formal tangent spaces

If $t \mapsto g_t$ is a family of Ricci-flat metrics, then

$$\frac{d}{dt} g_t \in \ker \mathrm{Hess}_g \mathcal{E} .$$

Thus if the space $\mathcal{M}_0(M)$ is smooth, then

$$T_g\left(\mathcal{M}_0(M)\right) \quad \subset \quad \left(\ker \mathrm{Hess}_g \mathcal{E}\right)$$

In the regular situation we have $=$ instead of $\subset$.

Formal tangent spaces

If $t \mapsto g_t$ is a family of Ricci-flat metrics, then

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Thus if the premoduli space $\mathcal{M}_0(M)/\,\mathrm{Diff}_{\mathrm{Id}}(M)$ is smooth, then

$$\begin{aligned} T_g \left(\mathcal{M}_0(M)/\,\mathrm{Diff}_{\mathrm{Id}}(M)\right) &\subset (\ker \mathrm{Hess}_g \mathcal{E})/\,T_g(\mathrm{Diff}_{\mathrm{Id}}(M)\cdot g \\ &= \ker \Delta^E \cap \mathrm{TT}^+ \end{aligned}$$

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In the regular situation we have $=$ instead of $\subset$.
The formal tangent space of $\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M)$ is

$$\begin{aligned} T_g^{\mathrm{form}}\left(\mathcal{M}_0(M)/\operatorname{Diff}_{\mathrm{Id}}(M)\right) &:= \ker(\operatorname{Hess}_g \mathcal{E})/T_g(\operatorname{Diff}_{\mathrm{Id}}(M) \cdot g \\ &= \ker \Delta^E \cap \mathrm{TT}^+ \end{aligned}$$

Corollary (…/Goto/Nordström/AKWW 2015)

If g is a structured Ricci-flat metric, then $\mathcal{M}_0(M)$ *is a manifold in a neighborhood of g and*

$$T_g^{\mathrm{form}}(\mathcal{M}_0(M)) = T_g(\mathcal{M}_0(M)).$$

Infinitesimal stability

Now, let again g be Riemannian.

$$\mathrm{TT}^+ := \{h \text{ symm. } (0,2)\text{-tensor} \mid \mathrm{tr}_g h = 0 \text{ and } \mathrm{div}_g h = 0\} \oplus \mathbb{R}g.$$

A Ricci-flat Riemannian metric g is called infinitesimally stable

$$:\Leftrightarrow (\mathrm{Hess}_g\, \mathcal{E})|_{\mathrm{TT}^+ \times \mathrm{TT}^+} \leq 0$$
$$\Leftrightarrow \Delta_E \text{ is positive semi-definite}$$

Interpretation of infinitesimal stability

There are no deformations $(g_t)_{t\in(-\epsilon,\epsilon)}$ with $g_0 = g$ and with $\frac{d}{dt}|_{t=0}\mathrm{scal}^{\,g_t} > 0$.

Structured metrics are infitesimally stable

Theorem (McKenzie Wang, Ind. Univ. Math. J. 1991)

Structured Ricci-flat metrics are infinitesimally stable.

More precisely: McK Wang assumes a parallel spinor on M, however can be easily extended to structured manifolds.

Proof of McKenzie Wang's theorem

The McKenzie Wang map $\mathcal{W}$ for $\varphi \in \Gamma_{\|}(Q, g)$ is given by

$$\bigodot^2 T^*Q \hookrightarrow \bigotimes^2 T^*Q \xrightarrow{\mathrm{cl}^g(\,\cdot\,,\varphi)\otimes\mathrm{id}} \$^g Q \otimes T^*Q$$

where $\mathrm{cl}(\alpha, \varphi) := \alpha^\# \cdot \varphi$ is the Clifford multiplication of 1-forms with spinors. If $\nabla\varphi = 0$, then

$$\begin{array}{ccc}
\Gamma(\bigodot^2 T^*Q) & \xrightarrow{\quad\mathcal{W}\quad} & \Gamma(\$^g Q \otimes T^*Q) \\
\Delta^E \big\downarrow & & \big\downarrow \left(D^{T^*Q}\right)^2 \\
\Gamma(\bigodot^2 T^*Q) & \xrightarrow{\quad\mathcal{W}\quad} & \Gamma(\$^g Q \otimes T^*Q)
\end{array}$$

commutes.
Here: $\Delta^E h = \nabla^*\nabla h - 2\mathring{R}h.$ $\qquad \mathring{R}h(X, Y) := h(R_{e_i,X}Y, e_i).$

Consequences of McKenzie Wang

Δ^E is the linearization of $g \mapsto 2\,\mathrm{Ric}^g$ in divergence, trace-free directions.

- $\mathrm{spec}(\Delta^E) \subset [0, \infty)$. Thus $g \in \mathcal{M}_{\|}(Q)$ is infinitesimally stable.
- $\ker \Delta^E \subset \ker \left(D^{T^*Q}\right)^2$.

Infinitesimal versus local stability

Infinitesimal stability:

Theorem (McKenzie Wang, Ind. Univ. Math. J. 1991)

Structured Ricci-flat metrics are infinitesimally stable.

Local stability:

Theorem (X. Dai, X. Wang, G. Wei, Inv. Math. 2005)

Structured Ricci-flat metrics g_0 with irreducible holonomy on a simply connected closed manifold are locally stable, i.e. there is no metric of positive scalar curvature in a neighborhood of g_0.

Historical details

Corollary (AKWW)

Local stability is true for arbitrary closed manifolds.

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Main results of Talk I

Recall $\mathcal{M}_{\|}(M) := \{g \in \mathcal{M}(M) \mid g \text{ structured}\}$
$\subset \mathcal{M}_0(M) := \{g \in \mathcal{M}(M) \mid \mathrm{Ric}^g = 0\}$

Theorem A (A.-Kröncke-Weiß-Witt)

Let M be a closed manifold whose universal covering is spin.

(1) *The premoduli space is $\mathcal{M}_{\|}(M)/\operatorname{Diff}_{\mathrm{Id}}(M)$ is a finite-dimensional smooth manifold.*

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(2) *The space is $\mathcal{M}_{\|}(M)$ is a smooth Frechet submanifold of $\mathcal{M}(M)$.*

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(3) *On $\mathcal{M}_{\|}(M)$, the map $g \mapsto \mathrm{Hol}(M, g)$ is locally constant up to conjugation.*

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(4) *On $\mathcal{M}_{\|}(M)$, the map $g \mapsto \dim \Gamma_{\|}(\Sigma_g M)$ is locally constant.*

If M is simply connected and if g_0 is a metric of irreducible holonomy, then the theorem was already known for metrics close to g_0.

Sketch of proof:

- After passing to a finite covering $\widehat{M} \to M$, we can decompose
$$\widehat{M} = M_1 \times \cdots \times M_r \times T^k$$
into irreducible factors and a torus. $\rightsquigarrow$ Part I A

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- Show that deformations "preserve" this structure. $\rightsquigarrow$ Kröncke's product formula.

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into irreducible factors and a torus. $\leadsto$ Part I A
- Show that deformations "preserve" this structure.
$\leadsto$ Kröncke's product formula.
- Use known results for the irreducible factors M_i.
This implies the result for $\widehat{M}$.
- Show that the statement survives under finite quotients.
Uses Theorem by Montgomery and Zippin.

Kröncke's product formula

Theorem (Kröncke)

If (M^m, g) and (N^n, h) are two closed infinitesimally stable Ricci-flat manifolds, then $(M \times N, g + h)$ is also infinitesimally stable.
Furthermore, on TT^+ we have

$$\begin{aligned}\ker(\Delta_E^{M\times N}) \cap \mathrm{TT}^+(M \times N) &= \left(\Gamma_{\|}(T^*M) \odot \Gamma_{\|}(T^*N)\right) \\ &\oplus \Big(\ker(\Delta_E^M) \cap \mathrm{TT}^+(M)\Big) \oplus \Big(\ker(\Delta_E^N) \cap \mathrm{TT}^+(N)\Big)\,.\end{aligned}$$

Corollary (Kröncke/AKWW)

The splitting

$$\widehat{M} = M_1 \times \cdots \times M_r \times T^k$$

is preserved under infinitesimal Ricci-flat deformations.

Comments on the irreducible case

The irreducible simply connected case is dealt in the literature with various techniques

- Kuranshi and Tian-Todorov theory in the Calabi-Yau case ($SU(n/2)$) and hyper-Kähler case ($\mathrm{Sp}(n/4)$).
- In terms of “positive” 3- resp. 4-forms in the case G_2 resp. Spin(7) (D. Joyce)

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Is there a conceptual argument, why the case-by-case reasoning in possible?
We will provide an infinitesimal version of such an argument in Part II A.

From $\widehat{M}$ to M

Theorem (Montgomery-Zippin)

Let H_0 be a compact subgroup of a Lie group G. Then there exists an open neighbourhood U of H_0 such that if H is a compact subgroup of G contained in U, then there exists $g \in G$ with $g^{-1}Hg \subset H_0$. Moreover, upon sufficiently shrinking U, g can be chosen in any neighbourhood of the identity of G.

This implies:
for any t_0 there is an $\epsilon > 0$ such that for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$ the group $\mathrm{Hol}(M, g_t)$ is conjugated to a subgroup H_t of $\mathrm{Hol}(M, g_{t_0})$.
We obtain

$$Hol_0(M, g_{t_0}) \subset H_t \subset \mathrm{Hol}(M, g_{t_0}) .$$

By "counting the connected components" we finally see that $H_t = \mathrm{Hol}(M, g_{t_0})$.

Talk I: Parallel spinors on Riemannian manifolds

A) Basic facts about Ricci-flat manifolds, special holonomy and spin geometry

B) Our results

C) Applications to the space of metrics g with $\mathrm{scal}^g \geq 0$
(D. Wraith, T. Schick)

Images/zahnrad-germany-and-japan.jp

Arxiv 1607.00657, T. Schick, D. Wraith, Non-negative versus positive scalar curvature

Application: Metrics of non-negative scalar curvature

Let M be a closed spin manifold.
$\mathrm{Scal}_{>0}(M) := \{g \in \mathcal{M}(M) \mid \mathrm{scal}^{\,g} > 0\}$
$\mathrm{Scal}_{\geq 0}(M) := \{g \in \mathcal{M}(M) \mid \mathrm{scal}^{\,g} \geq 0\}$

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Theorem (D. Wraith, Arxiv 1607.00657v1)

If g_t is a path in $\mathrm{Scal}_{\geq 0}(M)$ with $g_0 \in \mathrm{Scal}_{>0}(M)$, then $\ker D^V_{g(t)} = 0$ *for all t and all Dirac operators twisted with flat bundles V.*

Proof:

Assume $\ker D^V_{g_1} \ni \varphi \neq 0$. Then $\nabla^{g_1}\varphi = 0$ and $\mathrm{scal}^{\,g_1} = 0$. Apply Ricci flow to the path g_t. Let $\tau \in [0,1]$ be minimal such that $\mathrm{scal}^{\,g_t} = 0$ for $t \in [\tau, 1]$. Then $\mathrm{Ric}^{g_t} = 0$ for $t \in [\tau, 1]$ and $\mathrm{scal}^{\,g_{\tau-\epsilon}} > 0$. Thus g_τ has a parallel spinor. Thus cannot be deformed to $\mathrm{scal} > 0$. $\lightning$

Consequences

In fact one can extend these ideas:

- The space $\mathcal{M}_{||}(M)$ is closed and open in $\mathrm{Scal}_{\geq 0}(M)$ (unpublished)
- Wraith–Schick Arxiv 1607.00657:
  for $g \in \mathrm{Scal}_{\geq 0}(M) \setminus \mathcal{M}_{||}(M)$ the Mishchenko-Fomenko Dirac operator is invertible.
- Apply Ricci flow:
  $\mathrm{Scal}_{\geq 0}(M) \setminus \mathcal{M}_{||}(M)$ is homotopy equivalent to $\mathrm{Scal}_{>0}(M) \cup \big(\mathcal{M}_0(M) \setminus \mathcal{M}_{||}(M)\big)$.

$$\mathrm{Scal}_{\geq 0}(M) = \underbrace{\big(\mathrm{Scal}_{\geq 0}(M) \setminus \mathcal{M}_{||}(M)\big)}_{\text{open\&closed}} \sqcup \underbrace{\mathcal{M}_{||}(M)}_{\substack{\text{op.\&cl.}\\ \text{empty?}}}$$

“Corollary”

Every **known** *non-trivial element in* $\pi_k(\mathrm{Scal}_{>0}(M))$ *is also non-trivial in* $\pi_k(\mathrm{Scal}_{\geq 0}(M))$.

“Proof”

All known proofs showing that elements in $\pi_k(\mathrm{Scal}_{>0}(M))$ *are non-trivial use index theory. As* D^{MF} *is invertible not only on* $\mathrm{Scal}_{>0}(M)$ *but on* $\mathrm{Scal}_{\geq 0}(M) \setminus \mathcal{M}_{\|}(M)$ *the non-triviality extends to* $\mathrm{Scal}_{\geq 0}(M)$*).*

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Remark

- *I did* **not** *claim that* $\pi_k(\iota) : \pi_k(\mathrm{Scal}_{>0}(M)) \to \pi_k(\mathrm{Scal}_{\geq 0}(M))$ *is injective. The kernel of* $\pi_k(\iota)$ *might be large, but – to my knowledge – nobody has any method to detect or to show the existence of non-trivial elements in* $\ker \pi_k(\iota)$.

Talk II:

A) Parallel spinors on Lorentzian manifolds

(joint work with Klaus Kröncke and Olaf Müller)
Arxiv 1903.02064

Images/zahnrad-germany-and-japan.jpg

Spacelike hypersurfaces

Work by H. Baum, T. Leistner, A. Lischewski

If (N, h) is a Lorentzian manifold with a parallel spinor φ.
Then $h(V_\varphi, V_\varphi) \leq 0$, i.e. V_φ is causal.
We assume V_φ is lightlike.
Let M be a spacelike hypersurface of N with induced metric g and Weingarten map W.
On M we write

$$V_\varphi|_M = U_\varphi + u_\varphi \nu$$

U_φ tangential to M
ν future unit normal of M
If we “restrict” φ to M it satisfies the constraint equation

$$\begin{aligned} \nabla^M_X \varphi &= \frac{i}{2} W(X) \cdot \varphi, \qquad \forall X \in TM, \\ U_\varphi \cdot \varphi &= i u_\varphi \varphi, \end{aligned} \tag{CE}$$

The Cauchy problem for parallel spinors

Conversely, if we have a Riemannian manifold (M, g) with a non-trivial solution of

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then it extends to a Lorentzian metric on $M \times (-\epsilon, \epsilon)$ with a parallel spinor φ with V_φ lightlike.

Again: work by H. Baum, T. Leistner, A. Lischewski
Simplified by Julian Seipel (Master thesis, Regensburg), following ideas by P. Chrusciel

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Our Goal: Find solutions to (CE).

Machinery for solutions of the constraint equations

In Part I of the talk we discussed:

Theorem (Ammann, Kröncke, Weiß, Witt 2015)

The premoduli space

$$\mathcal{M}od_{||}(Q) := \mathcal{M}_{||}(Q)/\operatorname{Diff}_{\mathrm{Id}}(Q)$$

is a smooth manifold, and the map $\mathcal{M}od_{||}(Q) \to \mathbb{N}$, $[g] \mapsto \dim \Gamma_{||}(\$^g Q)$ *is locally constant.*

The parallel spinors from a vector bundle over

$$\Gamma_{\|} \to \mathcal{M}od_{\|}(Q)$$

of locally constant rank.
The bundle has

- a connection: given by work of Bourguignon—Gauduchon, Bär—Gauduchon—Moroianu and Müller—Nowaczyk.
- and a compatible metric: the L^2-metric

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- and a compatible metric: the L^2-metric

The connection in fact comes from a natural connection on the bundle

$$\coprod_{g\in\mathcal{M}(Q)} \Gamma(\$^g Q) \to \mathcal{M}(Q),$$

using the following (for us amazing) proposition:

Proposition (AKWW/AKM)

Along a divergence-free path of Ricci-flat metrics $(g_t)_{a \leq t \leq b}$ the parallel transport of a parallel spinor remains parallel.

We say that $(g_t)_{a \leq t \leq b}$ is divergence-free if

$$\mathrm{div}^{g_t}\left(\frac{d}{dt}g_t\right) = 0.$$

This means that this path of metrics is orthogonal to the orbits of $\mathrm{Diff}_{\mathrm{Id}}(Q)$.

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This proposition provides the conceptual reason promised in Part I B.

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This proposition provides the conceptual reason promised in Part I B.

Some comments on the proof

The following argument provides an infinitesimal version of the proposition.
Unfortunately it requires some work to obtain the full version out of it. In particular, this step requires the previous theorem.

Recall: Proof of McKenzie Wang's theorem

The McKenzie Wang map $\mathcal{W}$ for $\varphi \in \Gamma_{\|}(Q, g)$ is given by

$$\bigodot^2 T^*Q \hookrightarrow \bigotimes^2 T^*Q \xrightarrow{\mathrm{cl}^g(\,\cdot\,,\varphi)\otimes\mathrm{id}} \$^g Q \otimes T^*Q$$

where $\mathrm{cl}(\alpha, \varphi) := \alpha^{\#} \cdot \varphi$ is the Clifford multiplication of 1-forms with spinors. If $\nabla\varphi = 0$, then

$$\begin{array}{ccc} \Gamma(\bigodot^2 T^*Q) & \xrightarrow{\mathcal{W}} & \Gamma(\$^g Q \otimes T^*Q) \\ \Delta^E \downarrow & & \downarrow (D^{T^*Q})^2 \\ \Gamma(\bigodot^2 T^*Q) & \xrightarrow{\mathcal{W}} & \Gamma(\$^g Q \otimes T^*Q) \end{array}$$

commutes.
Here: $\Delta^E h = \nabla^*\nabla h - 2\mathring{R}h$. $\qquad \mathring{R}h(X, Y) := h(R_{e_i,X}Y, e_i)$.

Proof of the infinitesimal version of the proposition

$$
\begin{array}{ccc}
\Gamma(\bigodot^2 T^*Q) & \xrightarrow{\quad \mathcal{W} \quad} & \Gamma(\$^g Q \otimes T^*Q) \\
 & & \downarrow D^{T^*Q} \\
\Delta^E \downarrow & & \Gamma(\$^g Q \otimes T^*Q) \\
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\end{array}
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commutes.

Proof of the infinitesimal version of the proposition

On trace-free divergence-free tensors the diagram

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\Delta^E \downarrow & & \Gamma(\$^g Q \otimes T^*Q) \\
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\Gamma(\bigodot^2 T^*Q) & \xrightarrow{\quad \mathcal{W} \quad} & \Gamma(\$^g Q \otimes T^*Q)
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commutes. Here $\frac{d}{dg}\nabla\varphi$ is the “differential” of $\nabla\varphi \in \Gamma(\$^g Q \otimes T^*Q)$ with respect to the metric g.

$$\boxed{\Longrightarrow \quad \ker \tfrac{d}{dg}\nabla\varphi = \ker \Delta^E.}$$

Application to the Lorentzian problem

This structure provides solutions to the constraint equations:

$$\begin{aligned} \nabla^M_X \varphi &= \frac{i}{2} W(X) \cdot \varphi, \qquad \forall X \in TM, \\ U_\varphi \cdot \varphi &= i u_\varphi \varphi, \end{aligned} \tag{CE}$$

Theorem (Ammann, Kröncke, Müller 2019)

For any smooth curve $G : (a, b) \to \mathcal{M}od_{||}(Q)$ *and any smooth function* $F : (a, b) \to (0, \infty)$ *we obtain a solution of the constraint equation on* $M = Q \times (a, b)$ *with ...*

Theorem (Ammann, Kröncke, Müller 2019)

For any smooth curve $G : s \in (a,b) \to \mathcal{M}od_{||}(Q) \ni G_s$ and any smooth function $F : (a,b) \to (0,\infty)$ we obtain a solution of the constraint equation on $M = Q \times (a,b) \ni (x,s)$. The data on M are given as follows:

- *The metric Γ on M is $\Gamma = g_s + ds^2$ for a divergence-free family of metrics g_s with $[g_s] = G_s$*
- *The spinor is obtained as follows:*
  - *Choose a parallel spinor ψ_r on (Q, g_r) for some $r \in (a,b)$.*
  - *By parallel transport along $s \mapsto g_s$ choose a family $(\psi_s)_{s\in(a,b)}$ of spinors which are parallel on $(Q, g_s)_{s\in(a,b)}$.*
  - *View $\varphi := (F(s)\psi_s)_{s\in(a,b)}$ as a spinor over $M = Q \times (a,b)$.*
  - *This spinor φ satisfies the constraint equations.*

Summary of this part

For an interval I we get a map

$$C^\infty(I, \mathcal{M}od_{||}(Q)) \times C^\infty(I, \mathbb{R}_+) \to \mathcal{M}od_{||}^{\mathrm{Lor}}(Q \times I \times (-\epsilon, \epsilon))$$

Similarly closed curves $\mathcal{M}od_{||}(Q)$ yield Lorentzian metrics on $Q \times S^1 \times (-\epsilon, \epsilon)$ if a “closing” condition holds

Slogan: Curves of Riemannian special holononmy metrics on Q yield a Lorentzian special holonomy metric on a manifold N with $\dim N = \dim Q + 2$.

Talk II:

B) Topology of initial data sets with strict dominant energy condition

(work by Jonathan Glöckle)
Arxiv 1906.00099

Images/zahnrad-germany-and-japan.jpg

Topology of the space of Lorentzian initial data satisfying the dominant energy condition strictly

The dominant energy condition

Let h be a Lorentzian metric on N
Energy-momentum tensor or Einstein tensor

$$T^h := \mathrm{Ric}^h - \frac{1}{2}\mathrm{scal}^h h$$

We say that h satisfies the dominant energy condition in $x \in N$ if for all causal future oriented vectors $X, Y \in T_x N$:

$$T(X, Y) \geq 0. \qquad \text{(DEC)}$$

DEC on spacelike hypersurfaces

If M is a space-like hypersurface with induced metric g, and future-oriented unit normal, then we define:
Energy density $\rho := T^h(\nu,\nu) = \frac{1}{2}\left(\mathrm{scal}^{\,g} + (\mathrm{tr}\,W)^2 - \mathrm{tr}(W^2)\right)$
Momentum density $j := T^h(\nu,\,\cdot\,)|_{T_xM} = \mathrm{div}\,W - \mathrm{d}\,\mathrm{tr}\,W$
DEC for h implies $\rho \geq |j|$.

Definition

Let g be a Riemannian metric and W a g-symmetric endomorphism section. We say that (g, W) satisfies

- the dominant energy condition if $\rho \geq |j|$ (DEC)
- the strict dominant energy condition if $\rho > |j|$ (DEC_+)

$$
\begin{aligned}
\mathcal{I}(M) &:= \{(g,W) \mid g \in \mathcal{M}(M),\ W \in \mathrm{End}_{\mathrm{sym}}(TM)\}\,. \\
\mathcal{I}_{\geq 0}(M) &:= \{(g,W) \text{ satisfying (DEC)}\}\,. \\
\mathcal{I}_{>0}(M) &:= \{(g,W) \text{ satisfying } (\mathrm{DEC}_+)\}\,.
\end{aligned}
$$

The inclusion $\mathrm{Scal}_{>0}(M) \to \mathcal{I}_{>0}(M)$

$\mathcal{R}(M) \hookrightarrow \mathcal{I}(M)$, $g \mapsto (g, 0)$
$\mathrm{Scal}_{>0}(M) = \mathcal{R}(M) \cap \mathcal{I}_{>0}(M)$

Lemma

If $g \in \mathrm{Scal}_{>0}(M)$, then $(g, \lambda\,\mathrm{Id}) \in \mathcal{I}_{>0}(M)$ for all $\lambda \in \mathbb{R}$.

Lemma

If K is a compact set and $K \to \mathcal{I}(M)$, $k \mapsto (g_k, W_k) \in \mathcal{I}(M)$, then there is a $\lambda_\pm \in \mathbb{R}$ with $\pm\lambda_\pm \gg 0$ and such that for all $k \in K$

$$(g_k, W_k + \lambda_\pm\,\mathrm{Id}) \in \mathcal{I}_{>0}(M).$$

With such arguments it follows that the inclusion $\mathrm{Scal}_{>0}(M) \hookrightarrow \mathcal{I}_{>0}(M)$ is homotopic to a constant map. This leads to maps

$$\mathrm{Cone}\big(\mathrm{Scal}_{>0}(M)\big) \to \mathcal{I}_{>0}(M)$$

one map for $\lambda_+ \gg 0$ and one for $\lambda_- \ll 0$.
Glued together we get a map $\Sigma\big(\mathrm{Scal}_{>0}(M)\big) \to \mathcal{I}_{>0}(M)$.

The Dirac-Witten operator

Restrict the spinor $\$N$ from (N, h) to (M, g).
As Clifford module $\$N|_M$ is one or two copies of $\$M$.
However: scalar product on $\$N$ is indefinit, scalar product on $\$M$ positive definit.
The connections differ:

$$\nabla^N_X \varphi = \nabla^M_X \varphi - \frac{1}{2}\nu \cdot W(X) \cdot \varphi$$

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The connections differ:

$$\nabla^N_X \varphi = \nabla^M_X \varphi - \frac{1}{2}\nu \cdot W(X) \cdot \varphi$$

Dirac-Witten-Operator

$$D^{(g,W)}\varphi = \sum_{j=1}^{n} e_j \cdot \nabla^N_{e_j}\varphi$$

where $(e_1, \ldots, e_n)$ is a locally defined orthonormal frame of TM.

Theorem (Witten 1981, Parker-Taubes, Hijazi-Zhang, . . .)

$D^{(g,W)}$ is self-adjoint and invertible if $(g, W) \in \mathcal{I}_{>0}(M)$.

The Lorentzian α-index

Attention: $\mathrm{Cl}_{n,1}$-linear spinors instead of complex spinors

For every $(g, W) \in \mathcal{I}(M)$ we get an odd $\mathrm{Cl}_{n,1}$-linear self-adjoint Fredholm operator $D^{(g,W)}$.

$D^{(g,W)}$ is invertible if $(g, W) \in \mathcal{I}_{>0}(M)$.

For any $\Psi : S^{k+1} \to \mathcal{I}_{>0}(M)$ J. Glöckle constructs $\alpha_{\mathrm{Lor}}(\Psi) \in \mathrm{KO}^{-n-(k+1)-1,1}(\{*\}) \cong \mathrm{KO}^{-n-k-1}(\{*\})$.

Theorem (J. Glöckle 2019)

The diagram

$$\begin{array}{ccccc} \pi_k\Big(\mathrm{Scal}_{>0}(M)\Big) & \xrightarrow{\Sigma} & \pi_{k+1}\Big(\Sigma(\mathrm{Scal}_{>0}(M))\Big) & \longrightarrow & \pi_{k+1}\Big(\mathcal{I}_{>0}(M)\Big) \\ & \searrow^{\alpha_{\mathrm{Riem}}} & & \swarrow_{\alpha_{\mathrm{Lor}}} & \\ & & KO^{-n-k-1}(\{*\}) & & \end{array}$$

commutes.

Application

Non-triviality of many $\pi_k\big(\mathrm{Scal}_{>0}(M)\big)$ was shown by Crowley, Hanke, Steimle, Schick and Botvinnik, Ebert, Randal-Williams.
Strategy:

- Description of some $\Psi : S^k \to \mathrm{Scal}_{>0}(M)$.
- Show $\alpha_{\mathrm{Riem}}(\Psi) \neq 0$.

Corollary (Glöckle)

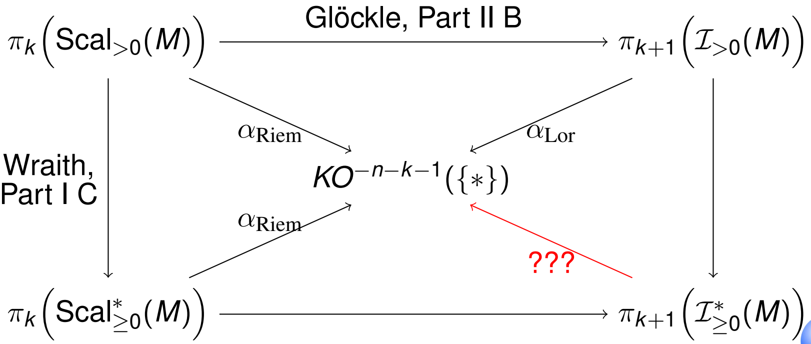
For each such non-trivial $\pi_k\big(\mathrm{Scal}_{>0}(M)\big)$ *we get a non-trivial* $\pi_{k+1}\big(\mathcal{I}_{>0}(M)\big)$.

Perspectives

$\mathrm{Scal}^*_{\geq 0}(M) := \mathrm{Scal}_{\geq 0}(M) \setminus \mathcal{M}_{\|}(M)$.

$\mathcal{I}^*_{\geq 0}(M) := \mathcal{I}_{\geq 0}(M) \setminus \{(g, W)$ having a spinor solving (CE)$\}$.

Recall from Part II A that (CE) were the constraint equations for the Cauchy problem for Lorentzian manifolds with parallel spinors.



Some related literature

- B. Ammann, K. Kröncke, H. Weiss, and F. Witt, *Holonomy rigidity for Ricci-flat metrics*, http://arxiv.org/abs/1512.07390
- B. Ammann, K. Kröncke, and O. Müller, *Construction of initial data sets for Lorentzian manifolds with lightlike parallel spinors*, http://arxiv.org/abs/1903.02064
- A. Besse, *Einstein manifolds*, Springer, Berlin, 1987.
- J. Glöckle, *Homotopy of the space of initial values satisfying the dominant energy condition strictly*, http://arxiv.org/abs/1906.00099
- R. Goto, *Moduli spaces of topological calibrations, Calabi-Yau, hyper-Kähler, G_2 and $\mathrm{Spin}(7)$ structures*, Internat. J. Math. **15** (2004), no. 3, 211–257.

Literature continued

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- J. NORDSTRÖM, *Ricci-flat deformations of metrics with exceptional holonomy*, Bull. Lond. Math. Soc. **45** (2013), 1004–1018. http://arxiv.org/abs/1008.0663.
- M. WANG, *Preserving parallel spinors under metric deformations*, Ind. Math. J. **40** (1991), 815–844.
- X. DAI, X. WANG, G. WEI, *On the Stability of Riemannian Manifold with Parallel Spinors*, Invent. Math. **161** (2005), 151–176.
- T. SCHICK, D. WRAITH, *Non-negative versus positive scalar curvature*, http://arxiv.org/abs/1607.00657

Thank you for your attention!

Images/zahnrad-germany-and-japan.jpg

And thanks to Kazuo for organizing this conference which supports the German-Japanese friendship

Besse group

For many results mentioned in the talk the best reference is the book “Einstein manifolds” by “Arthur Besse”. The authors of this book are:

Averous, Bérard-Bergery, Berger, Bourguignon, Derdzinski, DeTurck, Gauduchon, Hitchin, Houillot, Karcher, Kazdan, Koiso, Lafontaine, Pansu, Polombo, Thorpe, Valère

We also mention – possibly without citing explicitly – results by Cheeger, Gromoll, Calabi, Yau, de Rham, Tian, Todorov, Joyce, M. Wang, X. Dai, X. Wang, G. Wei, Nordström, H. Baum, C. Bär, A. Moroianu, Montgomery, Zippin, Wraith, Schick,. . .

I apologize for all those that I have omitted here.

Who proved the results in Part I?

- Most of the hard work was already done before us.
  If M is simply-connected and g irreducible, then the results were known by classical work (e.g. McKenzie Wang et al.) and work of Tian-Todorov, D. Joyce, R. Goto, J. Nordström
  E.g. Goto and Nordström proved the smoothness of the premoduli in this case.
- Reducible manifolds (i.e. products) were not so clear:
  H.-J. Hein could explain to me essential steps in the CY-case using Tian-Todorov theory.
- Our collaboration provided the missing steps.
- Key argument for our proof: Kröncke's product formula.
  Much simpler than the Tian-Todorov approach.
- Additional arguments for the non-simply-connected case.
  E.g. problematic: Full holonomy groups of structured Ricci-flat metrics are not classified in the reducible case.

Historical comments about stability

- Infinitesimal stability: This was proven by McK Wang (Indiana Univ. Math. J 1991) for manifolds with parallel spinors. The general case of structured Ricci-flat manifolds then follows using the fact that any closed structured Ricci-flat manifold is finitely covered by one with parallel spinor. Dai-Wang-Wei (Inv. Math. 2005) reprove this. Their argument for infinite fundamanetal group can be simplified using this finite covering.
- For local stability Dai-Wang-Wei require simple-connectedness (an assumption which is not hard to remove). Their proof however also uses irreducibility. The irreducibility is used at the step to argue that every formal deformation integrates to a deformation. In reducible metrics Dai-Wang-Wei's arguments only yield proofs for formal deformations of product type.
The word of "irreducibility" disappeared in the published version compared to the preprint without a generalized proof. Dai claims that a more general proof can be read off from some other results by the authors.