Self-adjoint codimension 2 boundary conditions for Dirac operators

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Classical boundary conditions for the Laplace operator

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary.
To model problems in physics or technical applications we often want to solve the Poisson problem

$$\Delta u = f$$

for a given function $f \in C^\infty(\Omega)$ or the eigenvalue problem

$$\Delta u = \lambda u$$

for $\lambda \in \mathbb{R}$.
Many solutions, and many solutions behave unphysically at the boundary.
$\rightsquigarrow$ We need boundary conditions!

The minimal and the maximal operator

We define

$$\Delta_c : C_c^\infty(\Omega) \to C_c^\infty(\Omega)$$

as a densely defined operator in $L^2(\Omega)$.
The minimal operator $\Delta_{\min}$ is the closure of Δ_c with respect to the graph norm.

$$L^2(\Omega) \supset H_0^2(\Omega) \xrightarrow{\Delta_{\min}} L^2(\Omega)$$

And the maximal operator is defined as its adjoint

$$\underbrace{\left\{u \in L^2(\Omega) \mid \Delta u \in L^2(\Omega)\right\}}_{:=\mathrm{dom}(\Delta_{\max}) \supsetneq H^2(\Omega)} \xrightarrow{\Delta_{\max}} L^2(\Omega).$$

The map

$$\begin{aligned} \tau = (\tau_D, \tau_N) : C^\infty(\overline{\Omega}) &\to C^\infty(\partial\Omega) \times C^\infty(\partial\Omega), \\ u &\mapsto (u|_{\partial\Omega}, (\partial_\nu u)|_{\partial\Omega}) \end{aligned}$$

extends to a trace map $\tau : \mathrm{dom}(\Delta_{\max}) \to \check{H}(\partial\Omega)$.

Symplectic structure on boundary data

Trace map $\tau : \mathrm{dom}(\Delta_{\max}) \to \check{H}(\partial\Omega)$.
Here $\check{H}(\partial\Omega)$ is a Sobolev type Hilbert space,

$$H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega) \subset \check{H}(\partial\Omega) \subset H^{-1/2}(\partial\Omega) \times H^{-3/2}(\partial\Omega)$$

on which a non-degenerate perfect symplectic pairing

$$\begin{array}{ccc} \check{H}(\partial\Omega) \times \check{H}(\partial\Omega) & \xrightarrow{B} & \mathbb{R} \\ \Big((v_1, w_1), (v_2, w_2)\Big) & \mapsto & \displaystyle\int_{\partial\Omega} (v_1 w_2 - v_2 w_1) \end{array}$$

is well-defined.
Green identity

$$\int_\Omega (\Delta u_1) u_2 - \int_\Omega u_1 \Delta u_2 = \pm B\big(\tau(u_1), \tau(u_2)\big).$$

Self-adjoint extensions

If $A \subset \check{H}(\partial\Omega)$ is a closed subspace, then

$$\underbrace{\tau^{-1}(A)}_{\subset \mathrm{dom}(\Delta_{\max})} \xrightarrow{\Delta_A := \Delta_{\max}|_{\tau^{-1}(A)}} L^2(\Omega)$$

is a closed extension.
Δ_A is self-adjoint iff A is a Lagrangian subspace of $(\check{H}(\partial\Omega), B)$.

Examples:
Dirichlet boundary conditions: $A := \{(0, w) \in \check{H}(\partial\Omega)\}$
Neumann boundary conditions: $A := \{(v, 0) \in \check{H}(\partial\Omega)\}$

Under some regularity assumptions:

- solution of the Poisson problem
- discrete, real spectrum
- eigenspaces finite-dimensional, spanned by smooth functions

Codimension 1 bdy cond. for Dirac operators

Now replace:

$\overline{\Omega} \subset \mathbb{R}^n$	$C^\infty(\overline{\Omega})$	Laplacian Δ
cpct. Riem. spin manfd. with boundary M	twisted spinors $C^\infty(M, V \otimes \Sigma M)$	twisted Dirac oper. $\not{D}$

$$\begin{aligned} \tau =: C^\infty(M, V \otimes \Sigma M) &\to C^\infty\big(\partial M, (V \otimes \Sigma M)|_{\partial M}\big), \\ u &\mapsto u|_{\partial\Omega} \end{aligned}$$

extends to a trace map $\tau : \mathrm{dom}(\not{D}_{\max}) \to \check{H}(\partial M)$.
Here $\check{H}(\partial\Omega)$ is a Sobolev type Hilbert space,

$$H^{1/2}\big(\partial M, (V \otimes \Sigma M)|_{\partial M}\big) \subset \check{H}(\partial M) \subset H^{-1/2}\big(\partial M, (V \otimes \Sigma M)|_{\partial M}\big)$$

with a skew-hermitian sesquilinear map
$B : \check{H}(\partial M) \times \check{H}(\partial M) \to \mathbb{C}$.

$$\int_M \langle \not{D}\varphi, \psi \rangle - \int_M \langle \varphi, \not{D}\psi \rangle = B\big(\tau(\varphi), \tau(\psi)\big).$$

Self-adjoint extensions for D

If $A \subset \check{H}(\partial M)$ is a closed subspace, then

$$\underbrace{\tau^{-1}(A)}_{\subset \mathrm{dom}(\not{D}_{\max})} \xrightarrow{\not{D}_A := \not{D}_{\max}|_{\tau^{-1}(A)}} L^2(M, V \otimes \Sigma M)$$

is a closed extension.
$\not{D}_A$ is self-adjoint iff A is a Lagrangian subspace of $(\check{H}(\partial M), B)$.

Under some regularity assumptions:

- discrete, real spectrum
- eigenspaces finite-dimensional, spanned by smooth sections
- $\not{D}_A$ is a Fredholm operator

$\rightsquigarrow$ Atiyah-Patodi-Singer index theorem.
See e.g. Bär-Ballmann arxiv 1101.1196.

The setting of our work

- Let (M, g) be a complete oriented Riemannian manifold, N a compact oriented submanifold of codimension 2.
- We assume that $M \setminus N$ is spin. Thus there is a complex spinor bundle $\Sigma \to M \setminus N$.
- Let $L \to M \setminus N$ be a flat hermitian line bundle. We get

$$\pi_1(M \setminus N) \to S^1 \subset \mathbb{C}$$

- $W := \Sigma \otimes L$ generalized spinor bundle over $M \setminus N$

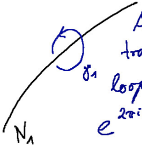
More general frameworks are possible which will not be discussed in this talk.

Monodromy

Monodromy $\alpha = (\alpha_1, \dots, \alpha_j)$.

$$N = \coprod_{j=1}^{\ell} N_j \qquad \text{decomposition into connected components}$$

Parallel transport in W around N_j is $e^{2\pi i \alpha_j}$.
$[\alpha_j] \in \mathbb{R}/\mathbb{Z}$ only depends on j.



Assume that parallel transport along a small loop around N_1 is $e^{2\pi i \alpha_1}$

Main examples

- M spin. Monodromy comes from L.
  Main subcase: L flat. Monodromy $\pi_1(M \setminus N) \to S^1$.
  Main subsubcase: N is a link in $\mathbb{S}^3$.

$$(S^1)^\ell \ni (\exp 2\pi i \alpha_j)_{j \in \{1,\ldots,\ell\}} \mapsto L_\alpha$$

- M is not spin, (more precisely: spin structure does not extend), N connected.
  Then monodromy only comes from Σ, $\alpha = 1/2 \mod \mathbb{Z}$.
  Main subcase: $L = \underline{\mathbb{C}}$
  Example: $M = \mathbb{C}P^{2r}$, $N = \mathbb{C}P^{2r-1}$.
  Fix $p \in M \setminus N$, solve $\not{D}\Psi = \psi_0 \delta_p$ on $M \setminus N$ with bdy cond.
  Expectation: If PMT would fail, we would get a map

$$S(\Sigma_p) \times \{\text{bdy cond}\} \to \{\text{non-zero spinors on } N\}.$$

Interesting applications?

Results by Ammann and Große

We obtain

- a bundle $S \to N$,
- a Hilbert space of sections $\check{H}(N, S)$,
- a skew-hermitian sesquilinear non-degenerate perfect pairing $B : \check{H}(N,S) \times \check{H}(N,S) \to \mathbb{C}$
- a trace map $\tau : \mathrm{dom}(\not{D}_{\max}) \to \check{H}(N, S)$

If $A \subset \check{H}(N, S)$ is a closed subspace, then

$$\underbrace{\tau^{-1}(A)}_{\subset \mathrm{dom}(\not{D}_{\max})} \xrightarrow{\not{D}_A := \not{D}_{\max}|_{\tau^{-1}(A)}} L^2(M, V \otimes \Sigma M)$$

is a closed extension.

$\mathrm{dom}(\not{D}_\emptyset) = \mathrm{dom}(\not{D}_{\min})$ and $\mathrm{dom}(\not{D}_{\check{H}(N,S)}) = \mathrm{dom}(\not{D}_{\max})$.

$\not{D}_A$ is self-adjoint iff A is a Lagrangian subspace of $(\check{H}(\partial M), B)$.

Some remarks

- In the case $\alpha \in \mathbb{Z}$ the submanifold N is invisible, i.e.

$$\not{D}_{\min} = \not{D}_{\max} = \not{D}^M$$

- τ is not the extension of a restriction map.
  In contrast we have: Suppose that $\varphi \in \mathrm{dom}(\not{D}_{\max})$ is bounded on a neighbourhood of N. Then $\varphi \in \mathrm{dom}(\not{D}_{\min})$.
- The bundle $S \to N$ has a Clifford multiplication

$$TM|_N \otimes S \to S$$

  Normal volume element: $\omega_{\mathrm{nor}} := e_1 \cdot e_2$ if (e_1, e_2) is a positively oriented orthonormal basis of the normal bundle.
  This gives a splitting

$$S = S^+ \oplus S^-$$

  into the $\pm i$-eigenspace bundles for the Clifford action of ω_{nor}.

Example: Portmann & Sok & Solovej

$\check{H}(N, S^+)$ and $\check{H}(N, S^-)$ are Lagrangian subspaces of $\check{H}(N, S)$.
PSS (2015–2018) considered the special case that $M = S^3$ is the round sphere and N is a link.
Electrons coupled magnetic fields.
Existence of harmonic spin^c-spinors yield statements of the type

> If our world is stable, then the fine structure constant $\hbar c/e^2$ has to satisfy some bounds.

Measurements: $\hbar c/e^2 = 137.03599968\ldots$.
A spectral flow argument yields harmonic spinors.
Question: How is this spectral flow related to classical link invariants?

2-dimensional model space

Assume $M = \mathbb{C} \ni z$, $N = \{0\}$, $\Sigma = \underline{\mathbb{C}^2} = \Sigma_+ \oplus \Sigma_-$
L flat bundle over $[\mathbb{C} : \{0\}]$, monodromy α
Then ω_{nor} is the standard volume element.

$$\not{D} = \not{D}^{\mathrm{nor}} = \sqrt{2} \begin{pmatrix} 0 & \overline{\partial} \\ -\partial & 0 \end{pmatrix}$$

$\frac{z^{-\alpha}}{|z|^{-\alpha}}$ represents a nowhere vanishing smooth section of L.
Ansatz:

$$\Phi^+_{\beta,\gamma} := \begin{pmatrix} z^\beta \overline{z}^\gamma \\ 0 \end{pmatrix}, \qquad \Phi^-_{\beta,\gamma} := \begin{pmatrix} 0 \\ z^\beta \overline{z}^\gamma \end{pmatrix}.$$

where β and γ over real numbers with $\beta - \gamma + \alpha \in \mathbb{Z}$.
$\Phi^\pm_{\beta,\gamma} \in L^2_{\mathrm{loc}}$ iff $\beta + \gamma > -1$

$$\not{D}\Phi^+_{\beta,\gamma} = -\sqrt{2}\beta\Phi^-_{\beta-1,\gamma}, \qquad \not{D}\Phi^-_{\beta,\gamma} = \sqrt{2}\gamma\Phi^+_{\beta,\gamma-1},$$

Lemma.

The condition that $\Phi^{\pm}_{\beta,\gamma} \in \mathrm{dom}(\not{D}_{\max})$ *is characterized as follows ("locally around* 0*").*

1. *Suppose* $\beta \neq 0$ *and* $\gamma \neq 0$*. Then* $\Phi^{\pm}_{\beta,\gamma} \in \mathrm{dom}(\not{D}_{\max})$ *if and only* $\beta + \gamma > 0$*.*
2. *Suppose* $\beta = 0$ *and* $\gamma \neq 0$*. Then* $\Phi^{+}_{0,\gamma} \in \mathrm{dom}(\not{D}_{\max})$ *if and only if* $\gamma > -1$*, and* $\Phi^{-}_{0,\gamma} \in \mathrm{dom}(\not{D}_{\max})$ *if and only if* $\gamma > 0$*.*
3. *Suppose* $\beta \neq 0$ *and* $\gamma = 0$*. Then* $\Phi^{+}_{\beta,0} \in \mathrm{dom}(\not{D}_{\max})$ *if and only if* $\beta > 0$*, and* $\Phi^{-}_{\beta,0} \in \mathrm{dom}(\not{D}_{\max})$ *if and only if* $\beta > -1$*.*
4. *Suppose* $\beta = \gamma = 0$*.* $\Phi^{\pm}_{0,0} \in \mathrm{dom}(\not{D}_{\max}) = \mathrm{dom}(\not{D}_{\min})$*.*

$\alpha \in (0,1)$: Then elements in $\mathrm{dom}(\not{D}_{\max})$ are of the form

$$\begin{pmatrix} \bar{z}^{\alpha-1}\varphi_+ \\ z^{-\alpha}\varphi_- \end{pmatrix} + \mathrm{dom}(\not{D}_{\min}).$$

Disclaimer

The project is still work in progress. Until now we have only written up in detail the case of a totally geodesic submanifold N and some similar assumptions, but we do not expect any modifications for the general case. Preprints are not yet available.

Summary

- We have a complete description of the self-adjoint extensions of twisted Dirac operators on manifolds $M^m \setminus N^{m-2}$.
- The space N is invisible, if the monodromy is trivial.
- Spinors in the domain of the maximal operator which are not in the minimal domain are not bounded (near the boundary). The trace map has to be modified considerably.
- Boundary data are given by elements in a Hilbert space $\check{H}$ of sections of a bundle $S \to N$. This space depends strongly on the monodromy.

Thanks...

... to Boris Botvinnik and Nikolai Saveliev for discussions about link invariants associated to these results.