Parallel spinors, Calabi–Yau manifolds and spinors

B. Ammann[1]

[1]Universität Regensburg, Germany

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The dominant energy condition

Let h be a Lorentzian metric on N
Energy-momentum tensor or Einstein tensor

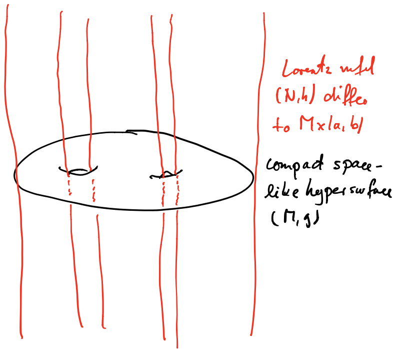
$$T^h := \mathrm{Ric}^h - \frac{1}{2}\mathrm{scal}^h h$$

We say that h satisfies the dominant energy condition in $x \in N$ if for all causal future oriented vectors $X, Y \in T_x N$:

$$T(X, Y) \geq 0. \qquad \text{(DEC)}$$

Physical interpretation (Einstein equation):

Non-negative mass density of matter fields.


Lorentz mfld
(N,h) diffeo
to M×]a,b[
compact space-
like hypersurface
(M,g)

DEC on spacelike hypersurfaces

If M is a space-like hypersurface with induced metric g, and future-oriented unit normal, then we define:
Energy density $\rho := T^h(\nu,\nu) = \frac{1}{2}\left(\mathrm{scal}^g + (\mathrm{tr}\, W)^2 - \mathrm{tr}(W^2)\right)$
Momentum density $j := T^h(\nu,\cdot)|_{T_xM} = \mathrm{div}\, W - \mathrm{d}\,\mathrm{tr}\, W$
DEC for h implies $\rho \geq |j|$.

Definition

Let g be a Riemannian metric and W a g-symmetric endomorphism section. We say that (g, W) satisfies

- the dominant energy condition if $\rho \geq |j|$ (DEC)

$$\mathcal{I}^{\geq}(M) := \{(g, W) \text{ satisfying (DEC)}\}.$$

- the strict dominant energy condition if $\rho > |j|$ ($\mathrm{DEC}_>$)

$$\mathcal{I}^{>}(M) := \{(g, W) \text{ satisfying } (\mathrm{DEC}_>)\}.$$

The inclusion $\mathcal{R}^{\geq}(M) \to \mathcal{I}^{\geq}(M)$

$$\mathcal{R}(M) \hookrightarrow \mathcal{I}(M),\ g \mapsto (g, 0)$$
$$\mathcal{R}^{\geq}(M) := \{g \in \mathcal{R} \mid \mathrm{scal}^{\,g} \geq 0\} = \mathcal{R}(M) \cap \mathcal{I}^{\geq}(M)$$
$$\mathcal{R}^{>}(M) := \{g \in \mathcal{R} \mid \mathrm{scal}^{\,g} > 0\} = \mathcal{R}(M) \cap \mathcal{I}^{>}(M)$$

Work by Jonathan Glöckle

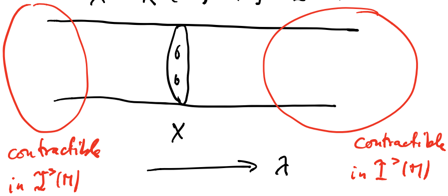
A lot is known about $\mathcal{I}^{>}(M)$.
In particular, we have $(g, \lambda g) \in \mathcal{I}^{>}(M)$ if

- $g \in \mathcal{R}^{>}(M)$ and $\lambda \in \mathbb{R}$, or
- $g \in \mathcal{R}^{\geq}(M)$ and $\lambda \in \mathbb{R} \setminus \{0\}$, or
- $g \in \mathcal{R}(M)$ and $|\lambda| \gg 0$.

We get a map $\mathrm{Susp}\big(\mathcal{R}^{>}(M)\big) \to \mathcal{I}^{>}(M)$.

$$\mathrm{Susp}\big(\mathcal{R}^{>}(M)\big) = \Big(\mathcal{R}^{>}(M) \times [-1,1]/M \times \{-1\}\Big)/M \times \{1\}.$$

$X = \mathcal{R}^{>}(M) \times \{0\} \subset \mathcal{I}^{>}(M)$



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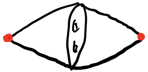

$\mathrm{Susp}(X)$

The Lorentzian α-index

For any $\Psi : S^{k+1} \to \mathcal{I}^{>}(M)$ J. Glöckle constructs $\alpha_{\mathrm{Lor}}(\Psi) \in \mathrm{KO}^{-n-(k+1)-1,1}(\{*\}) \cong \mathrm{KO}^{-n-k-1}(\{*\})$.

Theorem (J. Glöckle 2019)

The diagram

$$\begin{array}{ccccc} \pi_k\big(\mathcal{R}^{>}(M)\big) & \xrightarrow{\mathrm{Susp}} & \pi_{k+1}\big(\mathrm{Susp}(\mathcal{R}^{>}(M))\big) & \longrightarrow & \pi_{k+1}\big(\mathcal{I}^{>}(M)\big) \\ & \searrow^{\alpha_{Riem}} & & \swarrow_{\alpha_{Lor}} & \\ & & KO^{-n-k-1}(\{*\}) & & \end{array}$$

commutes.

Key technique in Glöckle's article: The Dirac-Witten operator

Literature: Witten 1981, Parker-Taubes, Hijazi-Zhang, ..., Glöckle 2019.

Restrict the spinor bundle ΣN from (N,h) to (M,g).

As spinor module $\Sigma N|_M$ is one or two copies of ΣM.

However:

scalar product $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$ on ΣN is indefinit (splitt signature),

scalar product $\langle \cdot\,,\,\cdot \rangle$ on ΣM positiv definit.

They are related by

$$\langle \varphi, \psi \rangle = \langle\!\langle \nu \cdot \varphi, \psi \rangle\!\rangle.$$

The connections differ:

$$\nabla^N_X \varphi = \nabla^M_X \varphi - \frac{1}{2}\nu \cdot W(X) \cdot \varphi$$

Dirac-Witten-Operator

$$D^{(g,W)}\varphi = \sum_{j=1}^{n} e_j \cdot \nabla^N_{e_j}\varphi$$

where $(e_1, \ldots, e_n)$ is a locally defined orthonormal frame of TM.

$D^{(g,W)}$ is self-adjoint and Fredholm.

Schrödinger-Lichnerowicz formula:

$$\left(D^{(g,W)}\right)^2 = (\nabla^N)^* \nabla^N + \frac{1}{2}(\rho - \nu \cdot j^\sharp \cdot),$$

The $*$ is taken on M with respect to $\langle \cdot, \cdot \rangle$.
Recall:
Energy density $\rho := T^h(\nu, \nu) = \frac{1}{2}\left(\mathrm{scal}^{\,g} + (\mathrm{tr} W)^2 - \mathrm{tr}(W^2)\right)$
Momentum density $j := T^h(\nu, \cdot)|_{T_x M} = \mathrm{div} W - \mathrm{d}\,\mathrm{tr} W$
DEC for h implies $\rho \geq |j|$.

This implies that $D^{(g,W)}$ is invertible if $(g, W) \in \mathcal{I}^{>}(M)$.
As a consequence Glöckle can use index theoretical methods.

Our goal

Question

Want to understand $\mathcal{I}^{\geq}(M)$.
Does this has similar properties as $\mathcal{I}^{>}(M)$?

Compare: Any non-trivial element in $\pi_k(\mathcal{R}^{>}(M))$, detected by index theory remains non-trivial in $\pi_k(\mathcal{R}^{\geq}(M))$.
In fact, suppose we have a map, $k > 0$,

$$g : S^k \to \mathcal{R}^{\geq}(M), \quad \alpha \mapsto g_\alpha,$$

and assume $g_\beta \in \mathcal{R}^{>}(M)$ for some $\beta \in S^k$.
Then Schick–Wraith showed that D^{g_α} is invertible for all $\alpha \in S^k$.
Important ingredients:

- if $\varphi \in \ker D^{g_\alpha}$, then $\nabla\varphi = 0$ ("the kernel case")
- Rigidity for metrics with parallel spinors (McKenzie Wang / Dai–Wang–Wei)

Alternative presentation of our question

In the following diagram we assume $k \geq 1$ and that the base point is g_0 resp. $(g_0, 0)$ where g_0 has positive scalar curvature.

$$\begin{array}{ccc} \pi_k\Big(\mathcal{R}^{>}(M)\Big) & \longrightarrow & \pi_{k+1}\Big(\mathcal{I}^{>}(M)\Big) \\ \downarrow & & \downarrow \\ \pi_k\Big(\mathcal{R}^{\geq}(M)\Big) & \longrightarrow & \pi_{k+1}\Big(\mathcal{I}^{\geq}(M)\Big) \end{array}$$

Index theoretically determined non-trivial homotopy groups survive in upper right and in lower left corner.
What about the lower right corner?

Analogous results for the Dirac–Witten operator?

Proposition (Ammann, Glöckle)

Assume that M is a connected closed spin manifold and $(g, W) \in \mathcal{I}^{\geq}(M)$.
We assume that $\varphi \in \ker D^{(g,W)} \setminus \{0\}$.
Then g, W, φ provides initial data for a Lorentzian manifold with a parallel spinor.

Example

If $(M, g, W) \subset (N, h)$ Lorentz manifold, $\Phi \not\equiv 0$ a parallel spinor.
Then $\nabla^N \Phi|_M = 0$, and thus $\Phi|_M \in \ker D^{(g,W)}$.

Moreover, we then have $\mathrm{Ric} = f\alpha \otimes \alpha$ for some lightlike α.
Thus $\mathrm{scal}^h = 0$ and $T = f\alpha \otimes \alpha$.

$$f \leq 0 \Leftrightarrow (g, W) \in \mathcal{I}^{\geq}(M).$$

Dirac currents

The Dirac current of a Lorentzian manifold (N, h) is the vector field V_φ with

$$h(X, V_\varphi) = -\langle\langle X \cdot \varphi, \varphi \rangle\rangle \quad \forall X \in TN$$

Then $h(V_\varphi, V_\varphi) \leq 0$, i.e. V_φ is causal.
If $V_\varphi(p)$ is lightlike, then $V_\varphi(p) \cdot \varphi(p) = 0$.

The Dirac current of a Riemannian manifold (M, g) is the vector field U_φ with

$$g(X, U_\varphi) = i\langle X \bullet \varphi, \varphi \rangle \quad \forall X \in TN$$

Unfortunately on $\Sigma N|_M$ two different Clifford multiplications are used in the literature:
$\cdot$ is given as the pullback of the Clifford multiplication on N.

$$X \bullet \varphi = i\nu \cdot X \cdot \varphi.$$

Then $V_\varphi|_M = -U_\varphi + u_\varphi \nu$ for some $u \in C^\infty(M)$.

Spacelike hypersurfaces

Work by H. Baum, T. Leistner, A. Lischewski

If (N, h) is a Lorentzian manifold with a parallel spinor φ.
As φ is parallel, V_φ is a parallel vector field.
Assuming M connected V_φ is either timelike or lightlike everywhere.

If V_φ is timelike, we locally have $N = M_0 \times \mathbb{R}$, $h = g - dt^2$.
The easier case.

Lightlike case

We assume V_φ is lightlike.

$$\nabla^N \varphi = 0,$$
$$V_\varphi \cdot \varphi = 0$$

If we "restrict" φ to M, these equations imply the constraint equations

$$\begin{aligned} \nabla^M_X \varphi &= \frac{i}{2} W(X) \bullet \varphi, \qquad \forall X \in TM, \\ U_\varphi \bullet \varphi &= i u_\varphi \varphi, \end{aligned} \tag{CE}$$

The Cauchy problem for parallel spinors

Conversely, if we have a Riemannian manifold (M, g) with a non-trivial solution of

$$\begin{aligned} \nabla^M_X \varphi &= \frac{i}{2} W(X) \bullet \varphi, \qquad \forall X \in TM, \\ U_\varphi \bullet \varphi &= i u_\varphi \varphi, \end{aligned} \tag{CE}$$

then it extends to a Lorentzian metric on $M \times (-\epsilon, \epsilon)$ with a parallel spinor φ with V_φ lightlike.

Again: work by H. Baum, T. Leistner, A. Lischewski
Simplified by Julian Seipel (Master thesis, Regensburg), following ideas by P. Chrusciel

Remark. In my last talk in Freiburg I explained:

Ammann–Kröncke–Müller proved:
For any family of metrics k_τ, $a < \tau < b$ on Q with a parallel spinors, we obtain a solutions to (CE) on

$$(M = Q \times (a,b), g = f_\tau^*(k_\tau) + d\tau^2). \qquad (*)$$

Leistner–Lischewski proved:
For any solution to (CE), M is locally isometric to a generalized version of (*).

Thus solutions to (CE) on m-dimensional manifolds are tightly related to families of metrics with parallel spinor on $(m-1)$-dimensional manifolds.

Some steps in the proof

Assume (DEC) and $\varphi \in \ker D^{(g,W)}$.
The Schrödinger-Lichnerowicz equation implies
Schrödinger-Lichnerowicz formula:

$$0 = \int_M \langle D^{(g,W)}\varphi, D^{(g,W)}\varphi\rangle \, d\mu^M$$
$$= \underbrace{\int_M \langle \nabla^N\varphi, \nabla^N\varphi\rangle \, d\mu^M}_{\geq 0} + \frac{1}{2}\underbrace{\int_M \langle (\rho - \nu \cdot j^\sharp \cdot)\varphi, \varphi\rangle \, d\mu^M}_{\geq 0},$$

This implies $\nabla^N\varphi = 0$, $\rho\varphi = \nu \cdot j^\sharp \cdot \varphi = -i\, j^\sharp \bullet \varphi$, and $\rho = |j|$.

The Lorentzian Dirac current V_φ is well-defined and ∇^N-parallel along M.
Thus V_φ is everywhere lightlike or everywhere timelike.

The case $\rho \neq 0$

In this case a calculation shows on $\{x \in M \mid \rho(x) \neq 0\}$:

$$U_\varphi = \frac{j^\sharp}{\rho} \|\varphi\|^2.$$

The definition of V_φ implies

$$V_\varphi = -U_\varphi + \|\varphi\|^2 \nu,$$

and thus V_φ is lightlike. This implies (CE).

The case $\rho \equiv 0$

Now $\rho = 0$ and $j = 0$. This means that the constraint equations for vaccuum Einstein equation are satisfied.
Thus we can choose N to be Ricci-flat (Choquet-Bruhat et al.).
We then can extend φ to a parallel section of ΣN.

Note: In this case φ might be spacelike or timelike.

Summary and outlook

Riemannian	Lorentzian IDS
$\mathrm{scal} \geq 0$	DEC
$\mathrm{scal} > 0$	DEC+
Dirac operator	Dirac–Witten operator
$\pi_k(\{\mathrm{scal} > 0\}) \xrightarrow{\alpha} KO_{m+k+1}(p)$	$\pi_{k+1}(\{\mathrm{scal} > 0\}) \xrightarrow{\alpha} KO_{m+k+1}(p)$
no $\mathrm{scal} > 0$ metric	no path from $(g, -Ng)$ to (g, Ng)
parallel spinor	initial data for Lorentzian manifolds with parallel spinors
smooth finite dim. moduli space	???
Wang rigidity	???
$\pi_k\{\mathrm{scal} \geq 0\}) \xrightarrow{\alpha} KO_{m+k+1}(p)$	???