Self-adjoint codimension 2 boundary conditions for Dirac operators

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Forschungsseminar Differentialgeometrie
Potsdam, June 25, 2020

The setting of this talk

- Let (M, g) be a complete oriented Riemannian manifold, N a compact oriented submanifold of codimension k.
- $[M : N] = (M \setminus N) \cup S_M N$ the blowup of M along N. Here $S_M N$ is the normal sphere bundle of N in M, $S_M N = \partial[M : N]$. The pull-back $\hat{g}|_p = (\pi^* g)|_p : T_p[M : N] \otimes T_p[M : N] \to \mathbb{R}$ is degenerate along the fibers of $S_M N$.

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  The pull-back $\hat{g}|_p = (\pi^* g)|_p : T_p[M : N] \otimes T_p[M : N] \to \mathbb{R}$ is degenerate along the fibers of $S_M N$.
- We assume that $M \setminus N$ is spin.
- There is a complex spinor bundle $\Sigma \to [M : N]$.
- Let $L \to [M : N]$ be a hermitian line bundle with ∇, whose curvature is a pull-back from M.
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More general frameworks are possible which will not be discussed in this talk.

Examples with different codimensions

- $\dim N = \dim M - 1$: Classical boundary problem.
  If N separates M in M_1 and M_2, then

  $$[M : N] = (M_1 \cup N) \amalg (M_2 \cup N).$$

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- $\dim N = \dim M - 2$. Monodromy $\alpha = (\alpha_1, \ldots, \alpha_j)$.

$$N = \coprod_{j=1}^{\ell} N_j$$

  Parallel transport in W around N_j is $e^{2\pi i \alpha_j}$.
  $[\alpha_j] \in \mathbb{R}/\mathbb{Z}$ only depends on j.
  Main objective of the talk.

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  $[\alpha_j] \in \mathbb{R}/\mathbb{Z}$ only depends on j.
  Main objective of the talk.
- $\dim N \leq \dim M - 3$.
  Then $L = \pi^*(\mathcal{L})$. No monodromy effects, N is "invisible".

Main examples

- M spin. Monodromy comes from L.
  Main subcase: L flat. Monodromy $\pi_1(M \setminus N) \to S^1$.
  Main subsubcase: N is a link in $\mathbb{S}^3$.

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- $(M \setminus N) \cup N_j$ is spin. Similar discussion close to N_j
- $(M \setminus N) \cup N_j$ is not spin, (more precisely: spin structure does not extend).
  Then monodromy only comes from Σ, $\alpha_j = 1/2 \mod \mathbb{Z}$.
  Main subcase: $L = \underline{\mathbb{C}}$
  Example: $M = \mathbb{C}P^{2r}$, $N = \mathbb{C}P^{2r-1}$.
  Fix $p \in M \setminus N$, solve $D\!\!\!/\Psi = \psi_0\delta_p$ on $M \setminus N$ with bdy cond.
  Expectation: If PMT would fail, we would get a map

$$S(\Sigma_p) \times \{\text{bdy cond}\} \to \{\text{non-zero spinors on } N\}.$$

Interesting applications?

Genesis of the project

Work by mathematical physicists for $M = \mathbb{S}^3$ or $M = \mathbb{R}^3$.
Electrons coupled magnetic fields.
Existence of harmonic spin^c-spinors yield statements of the type

> If our world is stable, then the fine structure constant $\hbar c/e^2$ has to satisfy some bounds.

Measurements: $\hbar c/e^2 = 137.03599968\ldots$. Why this?

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Examples of harmonic spin^c-spinors on $M = \mathbb{S}^3$ with distributional magnetic flux α along N yield smooth solutions on $\mathbb{R}^3$: smoothing of magnetic flux, conformal change.

Leads to link invariants, Hopf insulators (3-d topological insulators)

Some literature (incomplete!)

- Aharonov & Casher 1978: general description
- Loss & Yau (& Fröhlich) 1986: first examples of harmonic spinors, relation to “stablity of matter” and “estimates of the fine structure constant”
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- Deng& Wang & Sun & Duan: arxiv cond-mat 1612.01518 keywords: DNA, supramolecular chemistry, polymers, helium superfluid, spinor Bose-Einstein condensates, quantum chromodynamics, string theory, quantum Hall effects, topological insulators, Faddeev-Skyrme model, Hopfions,...
- Bi&Yan&Lu&Wang Phys. Rev. B 2017: Nodal-knot semimetals

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Interesting consequences for knot theory?

Interesting new boundary conditions for new applications?

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Disclaimer: Work **in progress.** Still sign mistakes, $\mathrm{l.o.t.}$-terms neglected etc.. Some parts will be sketchy.

Other mathematical literature?

The problem can be interpreted as a stratified space with strata of dimensions m and $m-2$.
Much literature, but our case does not seem to be covered.

- Albin & Gell-Redman 2016: incomplete edge space. Self-adjoint extensions, Fredholmness, index theory. This seems to fit. However, A&G-R require a spectral condition, called "Witt condition" which is in our case only satisfied for $\alpha \in \mathbb{Z}^\ell$.
- Mazzeo: has work prior to A&G-R on a blown-up version, seems to have gone into A&G-R
- Leichtnam & Mazzeo & Piazza
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It seems that we have to do the work ourselves.

Self-adjoint extensions. Again: the setting

(joint work in progress with Nadine Große, Freiburg)

- N a compact oriented submanifold of codimension 2 of M.
- $\pi : [M : N] \to M$ the blowup of M along N.
  $S_M N = \partial[M : N] = \pi^{-1} N$.
  $\hat{g} = \pi^* g : T_p[M : N] \otimes T_p[M : N] \to \mathbb{R}$ is degenerate along circle fibers of $S_M N \to N$
- $W \to [M : N]$ a suitable generalized $\hat{g}$ spinor bundle

$$N = \coprod_{j=1}^{\ell} N_j$$

Monodromy $\alpha = (\alpha_1, \dots, \alpha_\ell)$.
Parallel transport in W around N_j is $e^{2\pi i \alpha_j}$.

The associated Dirac operator $\not{D}$ is a formally self-adjoint 1st order differential operator.

Minimal and maximal closed extensions

Idea: Try to follow Bär-Ballmann

$C_c^\infty(W) := \{$sections of W with compact support in $[M:N]\}$
$C_{cc}^\infty(W) := \{$sections of W with compact support in $M \setminus N\}$

The minimal Dirac operator $\not{D}_{\min}$ is the Dirac operator whose domain is the closure of $C_{cc}^\infty(W)$ with respect to the graph norm

$$\|\varphi\|_D^2 := \|\varphi\|_{L^2}^2 + \|D\varphi\|_{L^2}^2.$$

$\not{D}_{\min}$ is symmetric.

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$D\!\!\!/_{\max} := D\!\!\!/_{\min}^*$, symmetry implies $\mathrm{dom}(D\!\!\!/_{\min}) \subset \mathrm{dom}(D\!\!\!/_{\max})$.

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$\not{D}_{\mathrm{max}} := \not{D}_{\mathrm{min}}^*$, symmetry implies $\mathrm{dom}(\not{D}_{\mathrm{min}}) \subset \mathrm{dom}(\not{D}_{\mathrm{max}})$.

Our Goal: Find domains $\mathcal{D}$ with $\mathrm{dom}(\not{D}_{\mathrm{min}}) \subset \mathcal{D} \subset \mathrm{dom}\, \not{D}_{\mathrm{max}}$ such that

$$\not{D}_{\mathrm{max}}|_{\mathcal{D}}$$

is self-adjoint.

First Surprise

$\mathrm{dom}(\not{D}_{\max})$ is **not** the closure of $C_c^\infty(W)$.

Problem: $\not{D} : C_c^\infty(W) \to C_c^\infty(W)$ not defined.

First Surprise

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Problem: $\not{D} : C_c^\infty(W) \to C_c^\infty(W)$ not defined.

Even worse: $\not{D}(\varphi|_{M\setminus N}) \notin L^2$, unless if φ is parallel along the circles of $S_M N \to N$.

What about $\overline{\{\varphi \in C_c^\infty(W) \mid \varphi \text{ parallel along circles}\}}$?

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Case 1: $\alpha_j \notin \mathbb{Z}$. Such φ vanish on N_j.

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Case 2: $\alpha_j \in \mathbb{Z}$**.** Then we have

$$\mathrm{dom}(\not{D}_{\max}) = \mathrm{dom}(\not{D}_{\min}),$$

i.e. $\not{D}_{\min}$ is self-adjoint!
Why this?

Thus: $C_c^\infty(W)$ **seems to be useless for us!**

The case $\alpha \in \mathbb{Z}^\ell$

In this case $W = \pi^*(\mathcal{W})$.

Lemma 1.

Let M be a complete manifold with generalized spinor bundle $\mathcal{W}$. Let $\mathrm{H}^1_{\not{D}}(M, \mathcal{W})$ be the completion of $C_c^\infty(M, \mathcal{W})$ w.r.t. the graph norm of $\not{D}$. If $N \subset M$ is (a compact submanifold) of codimension ≥ 2, then $C_c^\infty(M \setminus N, \mathcal{W})$ is dense in $\mathrm{H}^1_{\not{D}}(M, \mathcal{W})$.

Thus: "*N* is invisible."

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Proof.

Wlog codimension 2.
Let $\varphi \in C^\infty_c(M,\mathcal{W})$.
Take a logarithmic cut-off

$$\chi_{k,\epsilon}(x) := \begin{cases} 0 & \text{for } r(x) \leq e^{-k}\epsilon\,, \\ \frac{1}{k}\log\frac{r(x)\,e^k}{\epsilon} & \text{for } e^{-k}\epsilon \leq r(x) \leq \epsilon\,, \\ 1 & \text{for } r(x) \geq \epsilon\,. \end{cases} \tag{1}$$

Then

$$\left\|\nabla(\chi_{k,\epsilon}\varphi) - \nabla\varphi\right\|_{L^2} \leq C(\epsilon + \sqrt{k}). \tag{2}$$

For $\epsilon = k^{-1/2} \to 0$ we have $\chi_{k,\epsilon}\varphi \to \varphi$.

Some positive results (without proofs)

Lemma.
Suppose that $\varphi \in \mathrm{dom}(\not{D}_{\max})$ *is bounded on a neighbourhood of* N*. Then* $\varphi \in \mathrm{dom}(\not{D}_{\min})$*.*

Lemma.
Assume that the geometry of g *and* W *is bounded,* $\not{D}$ *coercive at infinity. Then on* $\mathrm{dom}(\not{D}_{\min})$ *the graph-norm for* $\not{D}$ *is equivalent to the classical* H^1*-norm, i.e. the graph norm for* ∇*.*

Lemma.
For an L^1_{loc}*-section* φ *of* W *we define* $D\varphi$ *in the distributional sense where as test functions we use the compactly supported smooth sections of* $W^* \otimes \bigwedge^n T^*M$*. Then* $\mathrm{dom}(\not{D}_{\max})$ *is the vector space of all* L^1_{loc}*-section of* W *for which* φ *and* $D\varphi$ *are in* L^2*.*

Abstract extension space

$$\check{Q} := \frac{\text{dom}\, \not{D}_{\max}}{\text{dom}\, \not{D}_{\min}}$$

abstract extension space with graph norm.
For $\varphi, \psi \in \text{dom}(\not{D}_{\max})$ we define

$$\check{b}([\varphi], [\psi]) := \int_{M \setminus N} \left(\langle \not{D}\varphi, \psi \rangle - \langle \varphi, \not{D}\psi \rangle \right) dv^g.$$

It is a well-defined, non-degenerate skew-hermitian form on $\check{Q}$.
Goals:
Identify this as Ȟ-sections of a bundle over N.
Show that the pairing is perfect.
$\{\text{self-adj. bdy cond.}\} \overset{1:1}{\longleftrightarrow} \{\text{Lagrangian subspaces of } (\check{Q}, \check{b})\}$

The normal volume element

Let (e_1, e_2) be a positively oriented orthornormal frame of the normal bundle $\nu_M N$ at p.
We define $\omega_{\text{nor}} := e_1 \cdot e_2 \in \text{End}(W_p)$.
Extend smoothly for p in neigborhood of N. Decompose into ω_{nor}-eigenspace bundles for eigenvalues $\pm i$.

$$W = W_+ \oplus W_-$$

$$\not{D} = \underbrace{\not{D}^{\text{nor}}}_{\text{odd}} + \underbrace{\partial_r \cdot \not{D}^N}_{\text{even}} + \text{l.o.t.}$$

Portman-Sok-Solovej boundary condition

Choose a sign $\epsilon_j \in \{\pm 1\}$ for each $j = 1, \dots, \ell$.
Close to N_j the boundary condition is

$$\mathcal{B} = \{\varphi \in \mathrm{dom}(\not{D}_{\max}) \mid (\omega_{\mathrm{nor}} + i\epsilon_j) \cdot \varphi \in \mathrm{dom}(\not{D}_{\max})\}.$$

Theorem 2 (PSS $\approx$ 2017).
This is a self-adjoint boundary condition in the case $M = \mathbb{S}^3$, N a link, L flat.

We extend this the whole setting, but there are many more self-adjoint extensions.

Continuity in α

Is the PSS boundary condition continuous in α?

The PSS boundary condition is

- continous for $\epsilon_j \alpha_j \nearrow 0 \mod \mathbb{Z}$,
- but non-continuous for $\epsilon_j \alpha_j \searrow 0 \mod \mathbb{Z}$.

General boundary conditions

The Ȟ-spaces have a both-sided regularity incontinuity at $\alpha_j \equiv 1/2 \mod 2$

Importance of continuity

Spectral flow arguments
Fredholm index is not constant at $\alpha_j \equiv 0 \mod \mathbb{Z}$.

2-dimensional model space

Assume $M = \mathbb{C} \ni z$, $N = \{0\}$, $\Sigma = \underline{\mathbb{C}^2} = \Sigma_+ \oplus \Sigma_-$
L flat bundle over $[\mathbb{C} : \{0\}]$, monodromy α
Then ω_{nor} is the standard volume element.

$$\not{D} = \not{D}^{\mathrm{nor}} = \sqrt{2}\begin{pmatrix} 0 & \overline{\partial} \\ -\partial & 0 \end{pmatrix}$$

$\frac{z^{-\alpha}}{|z|^{-\alpha}}$ represents a nowhere vanishing smooth section of L.
Ansatz:

$$\Phi^+_{\beta,\gamma} := \begin{pmatrix} z^\beta \overline{z}^\gamma \\ 0 \end{pmatrix}, \qquad \Phi^-_{\beta,\gamma} := \begin{pmatrix} 0 \\ z^\beta \overline{z}^\gamma \end{pmatrix}.$$

where β and γ over real numbers with $\beta - \gamma + \alpha \in \mathbb{Z}$.
$\Phi^\pm_{\beta,\gamma} \in L^2_{\mathrm{loc}}$ iff $\beta + \gamma > -1$

$$\not{D}\Phi^+_{\beta,\gamma} = -\sqrt{2}\beta\Phi^-_{\beta-1,\gamma}, \qquad \not{D}\Phi^-_{\beta,\gamma} = \sqrt{2}\gamma\Phi^+_{\beta,\gamma-1},$$

Lemma.

The condition that $\Phi^{\pm}_{\beta,\gamma} \in \mathrm{dom}(\not{D}_{\max})$ *is characterized as follows ("locally around* 0*").*

(1) *Suppose* $\beta \neq 0$ *and* $\gamma \neq 0$. *Then* $\Phi^{\pm}_{\beta,\gamma} \in \mathrm{dom}(\not{D}_{\max})$ *if and only* $\beta + \gamma > 0$.

(2) *Suppose* $\beta = 0$ *and* $\gamma \neq 0$. *Then* $\Phi^{+}_{0,\gamma} \in \mathrm{dom}(\not{D}_{\max})$ *if and only if* $\gamma > -1$, *and* $\Phi^{-}_{0,\gamma} \in \mathrm{dom}(\not{D}_{\max})$ *if and only if* $\gamma > 0$.

(3) *Suppose* $\beta \neq 0$ *and* $\gamma = 0$. *Then* $\Phi^{+}_{\beta,0} \in \mathrm{dom}(\not{D}_{\max})$ *if and only if* $\beta > 0$, *and* $\Phi^{-}_{\beta,0} \in \mathrm{dom}(\not{D}_{\max})$ *if and only if* $\beta > -1$.

(4) *Suppose* $\beta = \gamma = 0$. $\Phi^{\pm}_{0,0} \in \mathrm{dom}(\not{D}_{\max}) = \mathrm{dom}(\not{D}_{\min})$.

$\alpha \in (0, 1)$: Then elements in $\mathrm{dom}(\not{D}_{\max})$ are of the form

$$\begin{pmatrix} \bar{z}^{\alpha-1}\varphi_+ \\ z^{-\alpha}\varphi_- \end{pmatrix} + \mathrm{dom}(\not{D}_{\min}).$$

Higher dimensions: Extension map and Trace map

Now: For simplicity of presentation let N be connected.
Idea: The trace map is given by

$$\begin{array}{rcl} \mathcal{R} : \mathrm{dom}(\not{D}_{\max}) & \to & \Gamma(W|_{S_M N}) \\ \varphi & \mapsto & \lim_{r \searrow 0} \begin{pmatrix} r^{1-\alpha} & 0 \\ 0 & r^{\alpha} \end{pmatrix} \varphi|_{\partial U_r(N)} \end{array}$$

$$\begin{array}{rcl} \check{b}([\varphi], [\psi]) & \stackrel{\mathrm{def}}{=} & \int_{M \setminus N} \Big(\langle \not{D}\varphi, \psi \rangle - \langle \varphi, \not{D}\psi \rangle \Big) \, dv^g \\ & = & B\big(\mathcal{R}(\varphi), \mathcal{R}(\psi)\big) \end{array}$$

where $B(\Phi, \Psi) = \int_{S_M N} \langle \Phi, \partial_r \cdot \Psi \rangle \, d\mu$ and where μ is the S^1-equivariant measure on $S_M N$ with $\pi_* \mu = \mathrm{dvol}^N$.

Wish list

Extension operator

$$\mathcal{E} : \check{\mathrm{H}}_\alpha(W|_{S_M N}) := \mathrm{Image}(\mathcal{R}) \to \mathrm{dom}(\not{D}_{\max})$$

Wishes:

$$\begin{aligned} \mathcal{R} \circ \mathcal{E} &= \mathrm{Id} \\ \check{b}(\varphi, \mathcal{E}(\Psi)) &= B(\mathcal{R}(\varphi), \Psi) \end{aligned}$$

B is a perfect pairing on $\check{\mathrm{H}}_\alpha(W|_{S_M N})$.

To determine $\check{\mathrm{H}}_\alpha(W|_{S_M N})$ we have to consider

- S^1-equivariance
- regularity along N

Equivariance

Let $\alpha \in (0,1)$
$S^1 \subset \mathbb{C}$ acts on the S^1-principle bundle $S_M N \to N$:
$\rho : S^1 \to \mathrm{Diff}(S_M N)$.
Then $K := d\rho(i)$ a vector field on $S_M N$.
We define

$$\begin{aligned}
\Gamma_\alpha(W^+|_{S_M N}) &:= \left\{\Phi \in C^\infty(W^+|_{S_M N}) \text{ with } \nabla_K \Phi = i(1-\alpha)\Phi\right\} \\
\Gamma_\alpha(W^-|_{S_M N}) &:= \left\{\Phi \in C^\infty(W^-|_{S_M N}) \text{ with } \nabla_K \Phi = -i\alpha\Phi\right\} \\
\Gamma_\alpha(W|_{S_M N}) &:= \Gamma_\alpha(W^+|_{S_M N}) \oplus \Gamma_\alpha(W^-|_{S_M N})
\end{aligned}$$

$\Gamma_\alpha(W|_{S_M N})$ is the space of sections of a vector bundle over N.

Density and regularity

Relevance: Let $\Phi_\pm \in \Gamma_\alpha(W^\pm|_{S_M N})$.
Then
$$\chi(r)\left(r^{\alpha-1}\Phi_+ + r^{-\alpha}\Phi_-\right) \in \mathrm{dom}(\not{D}_{\max}).$$
Up to l.o.t. and $\nabla\chi$-terms it is in the kernel of the normal Dirac operator $\not{D}^{\mathrm{nor}}$.
$\Gamma_\alpha(W|_{S_M N})$ is dense in the Hilbert space $\check{\mathrm{H}}_\alpha(W|_{S_M N})$.
To explain the norm on the space we will discuss

- The canonical metric on the normal bundle
- The N-Dirac operator
- The $\check{\mathrm{H}}_\alpha$-spaces

Canonical metric on the normal bundle

To understand codimension 1 boundary conditions, one has to understand half-cylinders $N \times [0, \infty)$ first.
In fact, half cylinders are a special case of the (blown-up) canonical metric on the normal bundle.

Let $N \subset M$ be of codimension k. The canonical metric is a Riemannian metric on the total space of $\pi : \nu_M N \to N$ such that

- π is a Riemannian submersion,
- the horizontal spaces $\mathcal{H}_p$ are given by the connection on $\nu_M N \to N$,
- for $V \in \nu_M$ the vertical space in V is naturally isometric to $\nu_M N|_{\pi(V)}$.

The Dirac operator $\not{D}_0$ on $(\nu_M N, g_{\mathrm{can}})$ is our **model operator.**

The N-Dirac operator

The horizontal space also define a distribution $\mathcal{H}$ of codimension $k-1$ in $S_M N$.
For an onb $e_1, \ldots, e_{m-k}$ of $\mathcal{H}_p$ and $\varphi \in \Gamma(W|_{S_M N})$ we define the N-Dirac operator as

$$\left(\not{D}^N \varphi\right)|_p := -\sum_{j=1}^{m-k} \partial_r \cdot e_j \cdot \nabla_{e_j} \varphi.$$

Lemma.
The operator $\not{D}^N$ is an odd, formally self-adjoint, elliptic operator of Dirac type on N.

Back to our codimension 2 setting

On the model space we have

$$\not{D}_0 = \underbrace{\partial_r \cdot \nabla_r + \frac{K}{r} \cdot \nabla_{K/r}}_{\not{D}^{\mathrm{nor}}} + \partial_r \cdot \not{D}^N$$

$$= \partial_r \cdot \left(\nabla_r - \omega_{\mathrm{nor}} \cdot \nabla_{K/r} + \not{D}^N \right)$$

Note that

$$\left(\nabla_r - \omega_{\mathrm{nor}} \cdot \nabla_{K/r}\right)\left(r^{\alpha-1}\varphi_+\right) = 0$$

$$\left(\nabla_r - \omega_{\mathrm{nor}} \cdot \nabla_{K/r}\right)\left(r^{-\alpha}\varphi_-\right) = 0$$

Idea: Analyse this in a spectral decomposition for $\not{D}^N$
This will give us the Ȟ-space.

The $\check{\mathrm{H}}_\alpha$ spaces

"Theorem".

Let $\alpha \in (0,1)$. *We obtain a splitting*

$$\begin{aligned} \Gamma_\alpha(W|_{S_M N}) &= V_+ \oplus V_- \\ \check{\mathrm{H}}_\alpha(W|_{S_M N}) &= \overline{V_+}^{H^\beta} \oplus \overline{V_-}^{H^{-\beta}} \end{aligned}$$

where $\beta := \min\{\alpha, 1-\alpha\}$.
There is a surjective trace map $\mathcal{R} : \mathrm{dom}(\not{D}_{\max}) \to \check{\mathrm{H}}_\alpha(W|_{S_M N})$ *with kernel* $\mathrm{dom}(\not{D}_{\min})$ *and an injective extension map* $\mathcal{E} : \check{\mathrm{H}}_\alpha(W|_{S_M N}) \to \mathrm{dom}(\not{D}_{\max})$ *with*

$$\begin{aligned} \mathcal{R} \circ \mathcal{E} &= \mathrm{Id} \\ \check{b}(\varphi, \mathcal{E}(\Psi)) &= B(\mathcal{R}(\varphi), \Psi) \end{aligned}$$

B is a perfect pairing on $\check{\mathrm{H}}_\alpha(W|_{S_M N})$.

$V_- := \left\{ \Phi \in \Gamma_\alpha(W|_{S_M N}) \mid \Phi \text{ "extends" to a } \not{D}_0\text{-harmonic } L^2\text{-spinor} \right\}$

The Ansatz

Attention: $\not{D}^N$ anticommutes with ω_{nor}.
We assume $\not{D}^N \Phi = \lambda \Phi$, $\Phi = (\Phi_+, \Phi_-)$.
For $r \to \infty$: $\not{D}^N$ dominates, thus $L^2 \Leftrightarrow \lambda > 0$
For $r \to 0$: $\nabla_{K/r}$ dominates
Ansatz
We search for a solution asymptotic to $\exp(-\lambda r)\Phi$

$$\varphi = f_+(r)\Phi_+ + f_-(r)\Phi_-, \qquad f = (f_+, f_-)$$

$\not{D}_0 \varphi = 0$ then translates into

$$0 = f'(r) + \frac{1}{r}\begin{pmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{pmatrix} f(r) + \lambda f(r)$$

The asymptotics for $r \to 0$ of solutions of this ODE depend strongly on the sign of $\alpha - \frac{1}{2}$.

The $\check{\mathrm{H}}_\alpha$ spaces

For $\alpha \in (0, 1/2)$: for a smooth section $\Phi = (\Phi_+, \Phi_-)$ of $W|_{\mathbb{S}_M N}$

$$\|\Phi\|^2_{\check{\mathrm{H}}} := \|\Phi_+\|^2_{\mathrm{H}^{-\alpha}} + \|\Phi_-\|^2_{\mathrm{H}^{\alpha}}$$

For $\alpha \in (1/2, 1)$: for a smooth section $\Phi = (\Phi_+, \Phi_-)$ of $W|_{\mathbb{S}_M N}$

$$\|\Phi\|^2_{\check{\mathrm{H}}} := \|\Phi_+\|^2_{\mathrm{H}^{1-\alpha}} + \|\Phi_-\|^2_{\mathrm{H}^{\alpha-1}}$$

For $\alpha = 1/2$: the space V_- is spanned by the eigenspinors of $D\!\!\!/^N$ to the positive eigenvalues.

The extension map

On V_- it is obtained by solving the ODE backwards: from $r \to 0$ to $r \to \infty$.

$$V_- \to \mathrm{dom}(\not{D}_{\max})$$

What do we do with V_+? (for simplicity $\alpha \neq 1/2$)
Extend $\Phi \in V_+$ by

$$\mathcal{E}(\Phi) := r^{\beta-1} \exp\left(-|\not{D}^N| r\right) \Phi.$$

Then $\not{D}_0 \varphi \neq 0$, but the L^2-norm of $\not{D}_0 \varphi$ remains sufficiently well-controlled.

Why is it impossible to find an extension on a larger space $\widetilde{\mathsf{H}}$? Why is it impossible that $\mathsf{Image}\,\mathcal{R}$ is larger?

(Until now we only have seen arguments for $\check{\mathsf{H}}_\alpha \subseteq \mathsf{Image}\,\mathcal{R}$!)

Why is it impossible to find an extension on a larger space $\widetilde{\mathsf{H}}$? Why is it impossible that $\operatorname{Image}\mathcal{R}$ is larger?

(Until now we only have seen arguments for $\check{\mathsf{H}}_\alpha \subseteq \operatorname{Image}\mathcal{R}$!)

Answer: As we have found a space, on which B is a perfect pairing!
Consider the continuous map

$$\Psi \mapsto b(\varphi, \mathcal{E}(\Psi)) = B(\mathcal{R}(\varphi), \Psi)$$

Thus $B(\mathcal{R}(\varphi), \bullet) \in \widetilde{\mathsf{H}}^*$

$$\implies \quad \mathcal{R}(\varphi) \subseteq \widetilde{\mathsf{H}}^{*B} \subseteq \check{\mathsf{H}}^{*B} = \check{\mathsf{H}}.$$

So, if $\check{\mathsf{H}} \subsetneq \widetilde{\mathsf{H}}$ is a strict inclusion, then $\widetilde{\mathsf{H}}^{*B} \subsetneq \check{\mathsf{H}}$, thus we get a contradiction to $\mathcal{R} \circ \mathcal{E} = \mathrm{Id}$.