Parallel spinors, Calabi-Yau manifolds, and special holonomy

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Spin structures on semi-Riemannian manifolds

$n = k+m$ $\quad \mathbb{R}^{m,k} = \mathbb{R}^{m+k}$ with scalar product

$$\langle x, y \rangle = -\sum_{j=1-k}^{0} x^j y^j + \sum_{j=1}^{m} x^j y^j$$

indices run through $j \in \{1-k, 2-k, \dots, 0, 1, 2, 3, \dots, m\}$

$(E_i)_{i=1-k,\dots,m}$ is a generalized orthonormal basis

if

$$\langle E_i, E_j \rangle = \delta_{ij}\, \varepsilon_i \,;\quad \varepsilon_i = \begin{cases} (-1) & i \leq 0 \\ 1 & i > 0 \end{cases}.$$

The canonical basis $(e_i)_{i=1-k,\dots,m}$ is a gonb.

$$\begin{tikzcd}
\mathrm{Spin}_0(m,k) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\Gamma"] & SO_0(m,k) \arrow[d, hook] \\
\widetilde{Gl}_+(n,\mathbb{R}) \arrow[r] & Gl_+(n,\mathbb{R})
\end{tikzcd}$$

univ. covering for $m \geq 3$, $k \leq 1$

univ. covering for $n \geq 3$

Spin structures on oriented diff' mfds

M oriented mfd of dimension n.

$P_{Gl_+}(M) \to M$ $\quad Gl(n,\mathbb{R})$-princ bdl of pos. oriented frames

Def: A spin structure is a $\widetilde{Gl}_+(n,\mathbb{R})$-principal bdl $P_{\widetilde{Gl}_+}(M) \to M$ with a map

$$P_{\widetilde{Gl}_+}(M) \xrightarrow{\theta} P_{Gl_+}(M) \quad \text{s.th.}$$

$$\begin{tikzcd}
P_{\widetilde{Gl}_+}(M) \times \widetilde{Gl}_+(n,\mathbb{R}) \arrow[rr] \arrow[dd] \arrow[dr] & & P_{\widetilde{Gl}_+}(M) \arrow[dl] \arrow[dd] \\
& M & \\
P_{Gl_+}(M) \times Gl_+(n,\mathbb{R}) \arrow[rr] \arrow[ur] & & P_{Gl_+}(M) \arrow[ul]
\end{tikzcd}$$

commutes.

Now we fix a semi-Riem. metric g on M, and space- and time orientations. Def.?

Write

$$TM = \underbrace{\mathcal{T}}_{\text{oriented}}^{\text{timelike vectors}} \oplus \underbrace{\mathcal{S}}_{\text{oriented}}^{\text{spacelike vectors}}$$

Reduction from $P_{O(m,k)}(M)$ resp. $P_{SO(m,k)}(M)$ to an $SO_0(m,k)$-principle bdl $P_{SO_0(m,k)}(M)$

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as a metric version of a spin structure

$$\underset{Spin_0(m,k)}{\circlearrowleft} P_{spin_0}(M) := \Theta^{-1}(P_{SO_0}(M))$$

$$
\begin{tikzcd}
P_{Spin_0}(M) \times Spin_0(m,k) \arrow[rr] \arrow[dd] \arrow[dr] & & P_{Spin_0}(M) \arrow[dl] \arrow[dd] \\
& M & \\
P_{SO_0}(M) \times SO_0(m,k) \arrow[rr] \arrow[ur] & & P_{SO_0}(M) \arrow[ul]
\end{tikzcd}
$$

The Levi-Civita connection yields a connection 1-form on $P_{SO_0}(M)$ and $P_{Spin_0}(M)$.

Spinor modules for $\mathbb{R}^{m,k}$

$$(\mathbb{R}^{m,k})^{\otimes 0} := \mathbb{R} \ni \mathbb{1}$$

$$Cl_{m,k} = \bigoplus_{\ell=0}^{\infty} (\mathbb{R}^{m,k})^{\otimes \ell} \Big/ \langle X \otimes Y + Y \otimes X + 2g(X,Y)\mathbb{1} \mid X,Y \in \mathbb{R}^{m,k} \rangle$$

Write $\cdot$ instead of $\otimes$ on the quotient.

$$Spin(m,k) \mathrel{\hat{=}} \{ v_1 \cdot \ldots \cdot v_{2k} \mid g(v_i, v_i) = \pm 1 \ \forall i \} \subset Cl_{m,k}$$

$$Cl_{m,k} \otimes_{\mathbb{R}} \mathbb{C} \cong \begin{cases} \mathbb{C}^{2^r \times 2^r} & \text{if } m+k = 2r \\ \mathbb{C}^{2^r \times 2^r} \oplus \mathbb{C}^{2^r \times 2^r} & \text{if } m+k = 2r+1 \end{cases}$$

$$Cl_{m,k} \otimes_{\mathbb{R}} \mathbb{C} =: \mathbb{C}l_{m,k}$$

Thus: there are two / is one irreducible representations

$$\mathbb{C}l_{m,k} \xrightarrow[\sigma]{\cong} \mathrm{End}(\Sigma^{\pm}) \; ; \quad (X,\varphi) \longmapsto \sigma(X)(\varphi) =: X\cdot\varphi$$

$\Sigma \cong \mathbb{C}^{2^r}$, $\quad X \in \mathbb{R}^{m,k}$, $\quad \varphi \in \Sigma$, $\quad X\cdot\varphi \in \Sigma$

if $m+k$ is odd / even.

Scalar product on Σ?

Required: $\mathrm{Spin}_0(m,k)$-invariant

1^{st} case $k=0, m=n$.

$(\,,\,)$ any hermitian scalar product on Σ

$G =$ group generated by $e_1, e_2, \dots, e_m$ in $(Cl_{m,0})^*$

$$\langle \varphi, \psi \rangle := \sum_{g \in G} (g\cdot\varphi, g\cdot\psi)$$

$\Rightarrow \langle e_j\cdot\varphi, e_j\cdot\psi\rangle = \langle\varphi,\psi\rangle \quad \forall j$

$\Rightarrow \langle X\cdot\varphi, \psi\rangle = -\langle\varphi, X\cdot\psi\rangle \quad \forall X \in \mathbb{R}^m$

$\Rightarrow \mathfrak{spin}(m) = \mathrm{Lie}(\mathrm{Spin}(m))$ acts skew-hermitian

$\Rightarrow \mathrm{Spin}(m)$ acts isometrically

General case

$k \geq 0$, $n = m + k$.

$$\underbrace{e_{1-j} \cdot \varphi}_{\text{defines Clifford multipl. of } \mathbb{R}^{m,k}} := i \underbrace{e_{m+j} \cdot \varphi}_{\text{uses Clifford multiplication of } \mathbb{R}^{m+k,0}}$$

$\Rightarrow \quad \langle e_j \cdot \psi, e_j \cdot \varphi \rangle = \langle \psi, \varphi \rangle \quad \forall j = 1-k, 2-k, \ldots, m$

but not $Spin_0(m,k)$-invariant

$$\langle\langle \varphi, \psi \rangle\rangle := \langle e_{1-k} \cdot e_{2-k} \cdot \ldots \cdot e_0 \cdot \varphi, \psi \rangle$$

Then $\langle\langle X \cdot \varphi, \psi \rangle\rangle = (-1)^{k+1} \langle\langle \varphi, X \cdot \psi \rangle\rangle$

$\langle\!\langle\ ,\ \rangle\!\rangle$ is $Spin_0(m,k)$-invariant

but no longer positive definite.

split signature $(2^{r-1}, 2^{r-1})$

The spinor bundle

$$\Sigma M := P_{Spin_0}(M) \times_{Spin_0(m,k)} \Sigma$$

1) Clifford multiplication $T_pM \otimes \Sigma_p M \longrightarrow \Sigma_p M$

$$X, \varphi \longmapsto X \cdot \varphi$$

$$X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi + 2g(X,Y) \cdot \varphi = 0$$

2) Fiberwise hermitian product on Σ, split sign.

$$\langle\langle X\cdot\varphi,\psi\rangle\rangle = (-1)^{k+1}\langle\langle\varphi, X\cdot\psi\rangle\rangle$$

3) metric connection

(comes from connection-1-form on $P_{Spin_0}(M)$)

Our interests

- Riemannian ($k=0$) and Lorentzian ($k=1$) case
- Parallel sections of $\Sigma M \to M$ (=: parallel spinors)
- Dominant energy condition
- special holonomy, e.g. Calabi-Yau, G_2-holonomy, Spin(7)-holonomy

Parallel spinors and Ricci-curvature

Calculate $\sum \underset{\pm 1}{\underset{\shortparallel}{\varepsilon_i}}\, e_i \cdot R_{e_i, Y}\, \varphi = -\frac{1}{2} \mathrm{Ric}(Y) \cdot \varphi$

If φ is a parallel spinor, then $R_{X,Y} \varphi = 0$,

thus $\mathrm{Ric}(Y) \cdot \varphi = 0$

$$g(\mathrm{Ric}(Y), \mathrm{Ric}(Y))\, \varphi = -\mathrm{Ric}(Y) \cdot \overbrace{\mathrm{Ric}(Y) \cdot \varphi}^{=0} = 0$$

<u>$k=0$:</u> $\mathrm{Ric} = 0$

<u>$k=1$</u> $\forall Y$: $\mathrm{Ric}(Y)$ lightlike $\Rightarrow$ $\mathrm{rk}\, \mathrm{Ric} \leq 1$

$\Rightarrow \mathrm{Ric} = f \alpha \otimes \alpha$

$f \in C^\infty(M)$, $\alpha \in \Omega^1(M)$ lightlike

Parallel spinors on Riemannian mfds

$\boxed{m = \dim M}$

Idea: parallel spinors yield parallel tensors

$$\Sigma M \otimes_{\mathbb{C}} \Sigma M = \begin{cases} \Lambda^{\text{even}} T^*M \otimes_{\mathbb{R}} \mathbb{C} & \text{if } m \text{ odd} \\ \Lambda^* T^*M \otimes_{\mathbb{R}} \mathbb{C} & \text{if } m \text{ even} \end{cases}$$

$\Rightarrow$ Parallel spinors give parallel forms

Example 1 Kähler structures & pure spinors

$\varphi \in \Sigma_p M$, $\varphi \neq 0$

$$j_\varphi : T_pM \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \Sigma_p M$$
$$X \longmapsto X \cdot \varphi$$

$\ker j_\varphi$ is an isotropic space

$T_pM \otimes_{\mathbb{R}} \mathbb{C}$ is a cplx vector space with hermitian metric $\langle .,. \rangle \rightsquigarrow$ symplectic structure $\omega = \operatorname{Im}\langle .,. \rangle$

A cplx subspace $V \subset T_pM$ is isotropic

(def)

$\Leftrightarrow \quad \omega|_{V \times V} \equiv 0$

$\Leftrightarrow \quad V \perp \overline{V}$

$\Leftrightarrow \quad \forall z \in V: \quad \|\mathrm{Re}(z)\| = \|\mathrm{Im}(z)\|, \quad \mathrm{Re}(z) \perp \mathrm{Im}(z)$

<u>Pf of "ker j_φ isotropic":</u> Let $v, w \in \ker j_\varphi$. Then

$$0 = v \cdot \underbrace{w \cdot \varphi}_{=0} + w \cdot \underbrace{v \cdot \varphi}_{=0} = -2 \underbrace{\{v, w\}}_{\text{cplx bilin. ext.}} \varphi = -2 \underbrace{\overbrace{\langle v, \overline{w} \rangle}^{=0}}_{\text{hermitian ext.}} \varphi \quad \square$$

Thus we obtain a cplx structure on $\mathrm{Re}(V) = \mathrm{Im}(V) \subset T_pM$.

$$\cong V$$

Def.: We say that φ is pure if $\mathrm{Re}(\ker j_\varphi) = T_pM$.
(I.e. if $\ker j_\varphi$ is Lagrangian (= maximally isotropic))

Thus any pure section φ of $\Sigma M \to M$ defines an orthogonal almost cplx structure J on M.

Furthermore $\nabla\varphi = 0 \Rightarrow \nabla J = 0$
(kähler).

We thus have a Ricci-flat kähler metric

Subexample $n=4$ $\quad \Sigma_p M = \Sigma_p^+ M \oplus \Sigma_p^- M$

$\underset{\neq 0}{\varphi} \in \Sigma_p M$ is pure $\Longleftrightarrow \varphi \in \Sigma_p^+ M$ or $\varphi \in \Sigma_p^- M$

If $\varphi \in \Gamma(\Sigma M)$ is parallel, then either

- $\varphi \in \Gamma(\Sigma^+ M) \rightsquigarrow$ cplx structure
- $\varphi \in \Gamma(\Sigma^- M) \rightsquigarrow$ cplx structure

opposite orientation

- φ mixed, $\varphi = \underset{\neq 0}{\varphi_+} + \underset{\neq 0}{\varphi_-} \Rightarrow M$ flat, $M = \mathbb{R}^2 / \Gamma$

lattice

If we have a parallel pure spinor, we also obtain a parallel cplx volume form

$$\omega_{\mathbb{C}} \in \underbrace{\Lambda^{\frac{n}{2},0} M}_{\text{For kähler mflds: Curvature of this bdl} \cong \text{Ricci-form}} \cong \Lambda^{\frac{n}{2}}_{\mathbb{C}} (\underbrace{T^*M}_{\text{as holom. bdl}}) \Rightarrow \text{holomorphic}$$

<u>Def.</u> A Calabi-Yau mfd is Kähler mfd with a non-trivial holom. cplx volume form.

Yau's solution of the Calabi conj. $\Longrightarrow$ $\exists$ Ricci-flat Kähler metric in same Kähler class

Holonomy

$$\mathrm{Hol}(M,g,p) := \{ P_\gamma \mid \gamma \text{ loop based in } p \}$$

parallel transport $T_pM \circlearrowleft$ along γ

Back to our example:

$$\{\text{Parallel pure spinor}\} \overset{①}{\Longleftrightarrow} \mathrm{Hol}(M,g,p) \subset SU\left(\tfrac{n}{2}\right)$$

$$\overset{②}{\Longleftrightarrow} g \text{ Kähler \& Ricci flat}$$

Here $\Leftarrow$ means: it holds if $\pi_1(M) = \{1\}$.

But the following counterexamples exist:

To ① Enriques surface $K3 / \{id, f\} =: M^4$ $\quad f^2 = id$

The isometry f lifts to a map $\hat{f}$ on the spin structure, but $\hat{f} \circ \hat{f} = -id$. $\hat{A}(M) = 1$. M is not spin.

To ② $M = \mathbb{C}^2 / \Gamma$

Γ generated by

$(z,w) \mapsto (z+k, w)$, $k \in \mathbb{Z}^2$

$(z,w) \mapsto (iz, w+\ell)$, $\ell \in \mathbb{Z}$

$(z,w) \mapsto (z, w+i\ell)$, $\ell \in \mathbb{Z}$

$\mathrm{Hol}(M) = \left\langle \begin{pmatrix} i & \\ & 1 \end{pmatrix} \right\rangle$

Example 2: $n=7$

$$\Sigma = \Sigma_7^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

If $\varphi \in \Sigma_7^{\mathbb{R}} \setminus \{0\}$, then
$$\begin{aligned} \mathbb{R}^7 &\longrightarrow \varphi^{\perp} \subset \Sigma_7^{\mathbb{R}} \\ X &\longmapsto X \cdot \varphi \end{aligned}$$
is an isomorphism

For $i, j \in \{1, \dots, 7\}$, $i \neq j$, there is a $v_{ij} \in S^6 \subset \mathbb{R}^7$ with $e_i \cdot e_j \cdot \varphi = -v_{ij} \cdot \varphi$.

This defines an octonionic structure on $\mathbb{R} \oplus \mathbb{R}^7$.

$\alpha := - \sum_{i<j} e_i^b \wedge e_j^b \wedge V_{ij}^b$ is an associated 3-form

If $\varphi \in \Gamma(\Sigma^{\mathbb{R}} M^7)$ is parallel, then

$\alpha_\varphi \in \Omega^3(M)$ is a "positive" parallel 3-form.

$$\mathrm{Hol}(M, g, p) \subset G_2 = \mathrm{Aut}(\mathbb{O})$$

Example 3: $n = 8$

$$\Sigma = \Sigma_8^{\mathbb{R}} \otimes \mathbb{C} \; ; \; \Sigma_8^{\mathbb{R}} = \Sigma_8^{\mathbb{R}+} \oplus \Sigma^{\mathbb{R}}$$

$$\mathbb{P}\left(\Sigma_8^{\mathbb{R}\overset{-}{+}}\right) \rightarrowtail \Lambda^4 \mathbb{R}^8$$

image = "pos. 4-forms" with some other orient., comp. with metric

$$\mathrm{Hol}(M, g, p) \subset \mathrm{Spin}(7) \subset SO(8$$

two ways to embed, distinguished by orientation

Theorem: Let (M,g) be a complete Riem. spin mfd with a parallel spinor.

Then $\tilde{M} = \mathbb{R}^k \times N_1 \times \dots \times N_\ell$, where

isometric

each N_i is compact, simply connected, not a product

and $$\mathrm{Hol}(N_j) = \begin{cases} SU(\dim N_j/2) \\ Sp(\dim N_j/4) \\ G_2 \\ \mathrm{Spin}(7) \end{cases}$$