From Pythagorean triples to constant mean curvature surfaces

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Weierstraß representation for surfaces in $\mathbb{R}^3$

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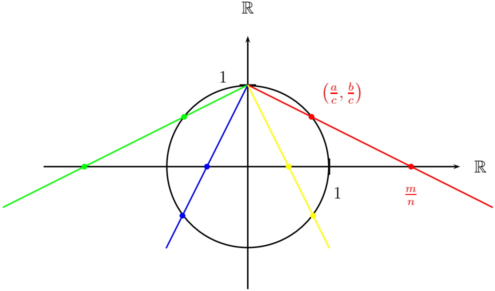
Weierstraß representation for surfaces in $\mathbb{R}^3$

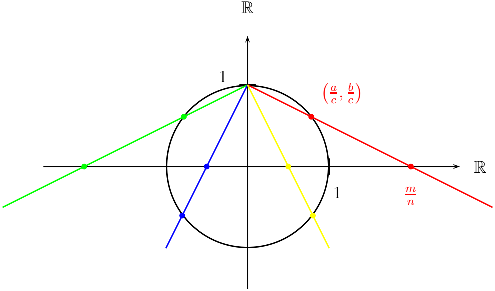
Pythagorean triples

$$a^2 + b^2 = c^2 \qquad a, b, c \in \mathbb{Z} \quad (*)$$

Equivalently: Solve $$\left(\frac{a}{c}, \frac{b}{c}\right) \in \left\{ v \in \mathbb{Q}^2 \,\middle|\, |v| = 1 \right\}$$

$$\overset{\text{stereogr. proj.}}{\longleftrightarrow} \text{ rational points on } \mathbb{R} \cup \{\infty\}$$





$$\left(\frac{a}{c}, \frac{b}{c}\right) = \left(\frac{2mn}{m^2+n^2}, \frac{m^2-n^2}{m^2+n^2}\right)$$

$$\begin{pmatrix} m \\ n \end{pmatrix} \mapsto \begin{pmatrix} a = 2mn \\ b = m^2 - n^2 \\ c = m^2 + n^2 \end{pmatrix}$$

$\mathbb{Z} \times \mathbb{Z} \rightarrow$ Solutions of (*) in $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

Complexification

Quadric $\quad Q = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{C}^3 \;\middle|\; a^2 + b^2 + c^2 = 0 \right\}$

$$\begin{array}{ccc} \begin{pmatrix} m \\ n \end{pmatrix} & \mapsto & \begin{pmatrix} m^2 - n^2 \\ i(m^2 + n^2) \\ 2mn \end{pmatrix} \\ \mathbb{C}^2 & \xrightarrow{2:1} & Q \\ \downarrow & & \downarrow \\ \mathbb{C}P^1 & \xrightarrow{1:1} & [Q] \end{array}$$

Conformal parametrizations of surfaces

Parametrization of a surface

$U \subset \mathbb{C}$ open, $(x, y) \in U$.
$F : U \to \mathbb{R}^3$ parametrization of a piece of a surface
$\frac{\partial F}{\partial z} = \frac{1}{2}\left(\frac{dF}{dx} - i\frac{dF}{dy}\right)$

F is conformal (=angle preserving)

$\Leftrightarrow \left|\frac{dF}{dx}\right| = \left|\frac{dF}{dy}\right|$ and $\frac{dF}{dx} \perp \frac{dF}{dy}$

$\Leftrightarrow \frac{\partial F}{\partial z} \in Q$

Weierstraß representation (≤1866)

$F : U \to \mathbb{R}^3$ $\qquad$ F conformal.
Find $\varphi_1, \varphi_2 : U \to \mathbb{C}$, such that

$$\frac{\partial F}{\partial z} = \begin{pmatrix} \varphi_1^2 - \varphi_2^2 \\ i(\varphi_1^2 + \varphi_2^2) \\ 2\varphi_1\varphi_2 \end{pmatrix}$$

$F(U)$ is a minimal surface (i.e. mean curvature $H = 0$)
$\Leftrightarrow \varphi_1$ and φ_2 are holomorphic functions.

Why is this important?

The equation $H = 0$ is a non-linear partial differential equation, thus a priori hard to solve.

$$\{\text{Solutions of } H = 0\} \longleftrightarrow$$

$$\{\text{Pairs } (\varphi_1, \varphi_2) \text{ of holomorphic functions}\}$$

Holomorphic functions are much easier to study.

Global Description

Under conformal coordinate transformations φ_1 and φ_2 behave the same way as square roots of 1-forms. Thus they are (half-)spinors.

Now let M be Riemann surface, conformally embedded (or immersed) into $\mathbb{R}^3$.

$$T^*M = \Sigma^+ M \otimes_{\mathbb{C}} \Sigma^+ M$$

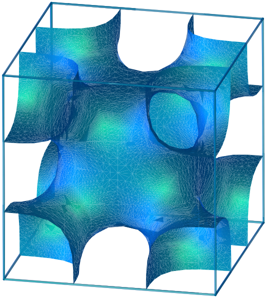
$$\Sigma^- M := \overline{\Sigma^+ M}$$

$$\Sigma M := \Sigma^+ M \oplus \Sigma^- M$$

$$\varphi := (\varphi_1, \bar{\varphi}_2) \in \Gamma(\Sigma M)$$

Dirac operator $D : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$

$$D \begin{pmatrix} \varphi_1 \\ \bar{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\partial \\ \bar{\partial} & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \bar{\varphi}_2 \end{pmatrix}.$$



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$$\left\{\begin{array}{l}\text{periodic minimal sufaces}\\ \text{with}\\ \text{fundamental domain } M\end{array}\right\} \Big/ \text{Translations}$$

$$\overset{1:1}{\longleftrightarrow}$$

$$\{\text{Pairs of holomorphic sections of } \Sigma^+ M\} \;/\; \pm 1$$

$$\overset{1:1}{\longleftrightarrow}$$

$$\{\text{Solutions of } D\varphi = 0\} \;/\; \pm 1$$

What about $H \neq 0$?

Kusner-Schmitt (1993/95/96):

$$\left\{\begin{array}{l}\text{Solutions of}\\ D\varphi = H\,|\varphi|^2\varphi\\ \text{on } M\end{array}\right\} /\pm 1$$

$$\overset{1:1}{\longleftrightarrow}$$

$$\left\{\begin{array}{l}\text{Conformal periodic immersions with}\\ \text{mean curvature function } H \text{ of } \widetilde{M} \text{ in } \mathbb{R}^3\\ \text{with branch points of even order}\end{array}\right\} \Big/ \text{Translations}$$

Modern presentations by Bär and Friedrich

Bär: 1997/98 (Special case previously by Trautman)
Assume N carries a parallel spinor ψ, e.g. $N = \mathbb{R}^3$, and that M is a hypersurface in N.
Then $\tilde{\varphi} := \psi|_M$ satisfies

$$\nabla_X \tilde{\varphi} = \frac{1}{2} W(X) \cdot \tilde{\varphi}, \quad |\tilde{\varphi}| = \mathrm{const} \overset{\mathrm{wlog}}{=} 1$$

for the induced metric $\tilde{g}$ on M.
Thus $D\!\!\!/\,\tilde{\varphi} = H\tilde{\varphi}$, $|\tilde{\varphi}| \equiv 1$.

Now suppose $\tilde{g} = f^4 g$. $\rightsquigarrow$ $D\!\!\!/^{\tilde{g}} = f^{-3} D\!\!\!/^{g} f$.
Then $\varphi := f\tilde{\varphi}$ satisfies

$$D\!\!\!/^{g} \varphi = H|\varphi|^2 \varphi.$$

Modern presentations by … and Friedrich

Friedrich: 1997/98, shortly afterwards
The energy-momentum tensor of φ will provide the Weingarten map W.
$\tilde{g} := |\varphi|^4 g$ and W satisfy:

a) Gauß equation: $\det W = K^{\tilde{g}}$

b) Codazzi equation: $(\nabla_X W)(Y) = (\nabla_Y W)(X)$

This allows a compatible immersion into $\mathbb{R}^3$.

The case $H = \mathrm{const}$ / Non-linear Dirac eigenvalues

Minimizing $\lambda_1^+(D\!\!\!/^g)\mathrm{vol}(M,g)^{1/n}$ in a conformal class

Let (M, g) be a closed Riemannian manifold, $n = \dim M$.
Let the spectrum of $D\!\!\!/^g$ be

$$-\infty \leftarrow \ldots \leq \lambda_1^-(g) < \underbrace{0 = \ldots 0}_{\dim \ker D\!\!\!/^g \text{ times}} < \lambda_1^+(g) \leq \ldots \to \infty$$

Lemma (Lott 1986, Ammann 2003)

$$\lambda_{\min}(M, [g]) := \inf_{\tilde{g} \in [g]} \lambda_1^+(g)\mathrm{vol}(M, g)^{1/n} > 0$$

Minimizing $\lambda_1^+(D\!\!\!/^g)\mathrm{vol}(M,g)^{1/n}$ in a conformal class

$$\lambda_{\min}(M,[g]) := \inf_{\tilde g\in[g]} \lambda_1^+(g)\mathrm{vol}(M,g)^{1/n} > 0$$

Theorem

If $\lambda_{\min}(M,[g]) < (n/2)\mathrm{vol}(\mathbb{S}^n)^{1/n}$, then the infimum is attained in a "generalized" metric.

The proof is similar to the solution of the Yamabe problem.
For $q = 2n/(n+1)$ one maximizes the functional

$$\psi \mapsto \mathcal{F}(\psi) = \frac{\int_M \langle D\!\!\!/^g\psi,\psi\rangle\,\mathrm{dvol}^g}{\|D\!\!\!/^g\psi\|^2_{L^q(M,g)}}$$

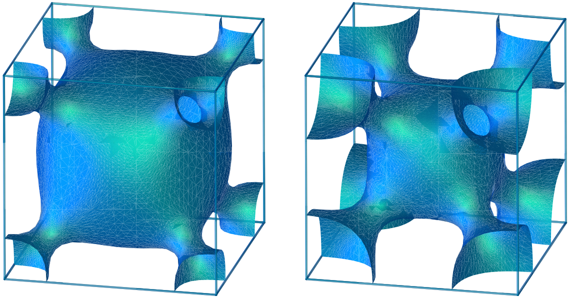
If ψ maximizes, the infimum is attained in $\tilde g := |D\!\!\!/\psi|^{4/(n+1)}g$.

The case $H = \text{const}$ cont'd

In this way one finds a maximizer ψ that satisfies

$$\not{D}\psi = \lambda_{\min}(M, g)|\psi|^{2/(n-1)}\psi$$

and for $n = 2$ this is the Weierstraß representation of a surface with $H \equiv \lambda_{\min}(M, g)$.



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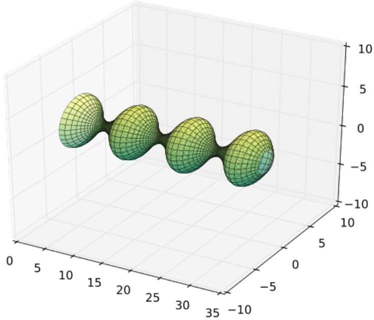
More pictures for $H = \mathrm{const}$

The unduloid – rectangular tori



We cannot show that this is the minimizer for the torus (conjecture: it is!), but if it is not, then there are very interesting other cmc surfaces based on the 2-torus.

Non-constant functions H

Suppose (M^2, g) is a closed Riemannian surface, $H : M \to \mathbb{R}^3$.
Is there a conformal map $F : M \to \mathbb{R}^3$ with mean curvature H?

Theorem (Ammann, Humbert, Ould Ahmedou)

If X is a conformal vector field on S^2 and $F : S^2 \to \mathbb{R}^3$ as above, then

$$\int_{S^2} \partial_X H \ \mathrm{dvol}^{F^* g_{\mathrm{eucl}}} = 0.$$

Consequence: Many functions, e.g. $H(x^1, x^2, x^3) = x^1$, are not a mean curvature!

Question: Does any mean curvature function on S^2 has at least 3 (or even 4) stationary points?

Four vertex theorem/Vierscheitelsatz: For $S^1 \hookrightarrow \mathbb{R}^2$ the function H has at least 4 stationary points.

For special functions existence results by M. Anderson and Tian Xu

Curvature in higher dimensions

Let M be a Riemannian manifold of arbitrary dimension $n \geq 2$.
$p \in M$; $Q \subset T_pM$ a 2-dimensional subspace.
For any $v \in Q$ take a geodesic $\gamma_v : [0, \epsilon) \to M$ with $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$.
The union of such curves is a (2-dimensional) surface S_Q in M.

Sectional curvature of M: $\mathrm{Sec}^M(Q, p) := K^{S_Q}(p)$ Gauß curvature of S_Q

For $v \in T_pM$, $|v| = 1$ we define

$$\mathrm{RIC}(v) := (n-1) \cdot (\text{average of } \mathrm{Sec}_Q \text{ over all } Q \ni v).$$

We say (M, g) is an Einstein manifold with Einstein constant λ if

$$\mathrm{RIC} \equiv \lambda$$

Higher dimensions: manifolds with parallel spinors

Recall: Bär's method for the Weierstraß representation requires a parallel spinor on A Riemannian manifold N
Classical case $N = \mathbb{R}^3$.
Manifolds with a parallel spinor are Ricci-flat $\mathrm{RIC} \equiv 0$.
They are *structured* Ricci-flat.

Thus: If $M \hookrightarrow N$ is a conformal (oriented) embedding of a hypersurface into such an N, then restriction of this spinor gives a solution to

$$D\psi = H|\psi|^{2/(n-1)}\psi \text{ on } M.$$

Is there a converse?
For general $n \geq 2$ and $N = \mathbb{R}^{n+1}$ very rarely!

Hypersurfaces in structured Ricci-flat manifolds

B. A.– A. Moroianu – S. Moroianu (2013):
A generalized Killing vector is a spinor solution ψ to

$$\nabla_X \psi = A(X) \cdot \psi, \quad \forall X \in TM$$

for some $A \in \mathrm{End}(TM)$.

Results in general dimensions $n \geq 2$

1.) If $M^n \subset N^{n+1}$ with N structured Ricci-flat, then there is a generalized Killing spinor on M wrt the induced metric $\tilde{g}$
Bär– "Extrinsic ..." or even Trautman

2.) If there is a generalized Killing spinor on $(M, \tilde{g})$, we can get a suitable N formally (as a power series).

3.) The power series converges if $\tilde{g}$ and A are real-analytic; we have examples of non-convergent series with the analyticity assumption.

3-dim. hypersurfaces in structured Ricci-flat 4-mnfds

There is a converse for $n = 3$!

Results in dimension $n = 3$

4.) Let ψ be a solution of

$$\not{D}^g \psi = H|\psi|\psi,$$

then $\tilde{\psi} = |\psi|^{-1}\psi$ is a generalized Killing spinor for $\tilde{g} = |\psi|^2 g$.
Thus for any (real-)analytic conformal class on a closed manifold M^3 we get a conformal embedding of constant mean curvature into a structured Ricci-flat N away from $\psi^{-1}(0)$.
5.) If the conformal class is not analytic, no such embedding exists.

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